Higher order corrections to the Grimus-Stockinger formula.

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Grimus-Stockinger theorem

Let us consider the integral:

$$\mathcal{J}(\boldsymbol{R}) = \int \frac{d^3q}{(2\pi)^3} \frac{e^{-i(\mathbf{q}\cdot\boldsymbol{R})}\Phi(\mathbf{q})}{(\mathbf{q}^2 - \kappa^2 - i0)}$$

According to the Grimus-Stockinger theorem, for a function $\Phi(\mathbf{q}) \in \mathcal{C}^3$, the asymptotic behaviour when $|\boldsymbol{R}| \to \infty$ is:

$$\mathcal{J}(\boldsymbol{R}) \approx \frac{e^{i\kappa R}}{4\pi R}\Phi\left(-\kappa\mathbf{n}\right)\left[1 + \mathcal{O}(R^{-1/2})\right], \quad \boldsymbol{R} = R\mathbf{n},$$

In order to identify the true expansion paramater, let's consider the higher order corrections with respect to R^{-1}, supposing that $\Phi(\mathbf{q})$ is represented by an absolutely convergent Fourier-transform.

By using the Fourier-image

$$\Phi(\mathbf{q}) = \int d^3\mathrm{x} e^{i(\mathbf{q}\cdot\mathbf{x})}\varphi(\mathbf{x})$$

and interchaning the integration order, we have

$$\mathcal{J}(\boldsymbol{R}) = \int \frac{d^3\mathrm{q}}{(2\pi)^3} \frac{e^{-i(\mathbf{q}\cdot\boldsymbol{R})}\Phi(\mathbf{q})}{(\mathbf{q}^2 - \kappa^2 - i0)} = \int d^3\mathrm{x} \frac{e^{i\kappa|\boldsymbol{R}-\mathbf{x}|}\varphi(\mathbf{x})}{4\pi|\boldsymbol{R} - \mathbf{x}|},$$

with the help of the spherical wave's expression as a free Green function

$$\frac{e^{i\kappa|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x} - \mathbf{y}|} = \int \frac{d^3\mathrm{q}}{(2\pi)^3} \frac{e^{i(\mathbf{q}\cdot(\mathbf{x}-\mathbf{y}))}}{(\mathbf{q}^2 - \kappa^2 - i0)} =< \mathbf{x}| \frac{1}{\left(\mathsf{P}^2 - \kappa^2 - i0\right)}|\mathbf{y} > .$$

Substituting the expansion

$$|\boldsymbol{R}-\mathbf{x}| = R\left[1-2\frac{(\mathbf{n}\cdot\mathbf{x})}{R}+\frac{\mathbf{x}^2}{R^2}\right]^{1/2} = R-(\mathbf{n}\cdot\mathbf{x})+\frac{\mathbf{x}^2-(\mathbf{n}\cdot\mathbf{x})^2}{2R}+O(R^{-2}),$$

one obtains:

$$\mathcal{J}(\boldsymbol{R}) = \frac{e^{i\kappa R}}{4\pi R}\int d^3\mathrm{x}e^{-i\kappa(\mathbf{n}\cdot\mathbf{x})}\varphi(\mathbf{x})\left[1+\frac{(\mathbf{n}\cdot\mathbf{x})}{R}+\frac{i\kappa}{2R}\left(\mathbf{x}^2-(\mathbf{n}\cdot\mathbf{x})^2\right)+O(R^{-2})\right],$$

which may be rewritten using the relations like

$$\int d^3\mathrm{x}e^{i(\mathbf{q}\cdot\mathbf{x})}\mathbf{x}\varphi(\mathbf{x}) = -i\boldsymbol{\nabla}_q\Phi(\mathbf{q}), \quad \text{and so on, as}$$

$$\mathcal{J}(\boldsymbol{R}) = \frac{e^{i\kappa R}}{4\pi R}\left[1-\frac{i}{R}(\mathbf{n}\cdot\boldsymbol{\nabla}_q)+\frac{i\kappa}{2R}\left((\mathbf{n}\cdot\boldsymbol{\nabla}_q)^2-\boldsymbol{\nabla}_q^2\right)+O(R^{-2})\right]\Phi(\mathbf{q})\Big|_{\mathbf{q}=-\kappa\mathbf{n}},$$

$$\text{where } (\mathbf{n}\cdot\boldsymbol{\nabla}_q)\Phi(\mathbf{q})\Big|_{\mathbf{q}=-\kappa\mathbf{n}} \Longrightarrow -\partial_\kappa\Phi(-\kappa\mathbf{n}).$$

In the case of a quadratic form of $\mathbf{q}$:

$$\Phi(\mathbf{q}) \Longrightarrow \mathcal{H}(\zeta), \quad \zeta = (\mathbf{q}\boldsymbol{A}^{-1}\mathbf{q}),$$

$$\left[1 - \frac{i}{R}(\mathbf{n}\cdot\boldsymbol{\nabla}_q) + \frac{i\kappa}{2R}\left((\mathbf{n}\cdot\boldsymbol{\nabla}_q)^2 - \boldsymbol{\nabla}_q^2\right)\right]\Phi(\mathbf{q})\bigg|_{\mathbf{q}=-\kappa\mathbf{n}} \Longrightarrow$$

$$\Longrightarrow \left[1 + \frac{i\kappa}{R}\left(3\overline{\alpha}(\mathbf{n}) - Tr\{\boldsymbol{A}^{-1}\}\right)\partial_\zeta - \frac{i2\kappa^3}{R}\left(\overline{\alpha^2}(\mathbf{n}) - \overline{\alpha}^2(\mathbf{n})\right)\partial^2_{\zeta^2}\right]\mathcal{H}(\zeta)\bigg|_{\zeta=\kappa^2\overline{\alpha}(\mathbf{n})},$$

where

$$\overline{\alpha}(\mathbf{n}) = \left(\mathbf{n}\boldsymbol{A}^{-1}\mathbf{n}\right), \quad \overline{\alpha^2}(\mathbf{n}) = \left(\mathbf{n}\boldsymbol{A}^{-2}\mathbf{n}\right).$$

For a Gaussian wave packet with the coordinate width σ_x, one has $\boldsymbol{A} \sim \sigma_x^{-2},\ \mathcal{H}(\zeta) = e^{-\zeta/4}$, so

$$\mathcal{J}(\boldsymbol{R}) = \frac{e^{i\kappa R}}{4\pi R} e^{-\kappa^2 \overline{\alpha}^2(\mathbf{n})/4} \left[1 - \frac{i\kappa}{4R} \left(3\overline{\alpha}(\mathbf{n}) - Tr\{\boldsymbol{A}^{-1}\} \right) - \frac{i\kappa^3}{8R} \left(\overline{\alpha^2}(\mathbf{n}) - \overline{\alpha}^2(\mathbf{n}) \right) \right],$$

or, with the same precision:

$$\mathcal{J}(\boldsymbol{R}) = \frac{e^{i\Theta(\boldsymbol{R})}}{4\pi R} e^{-\kappa^2 \overline{\alpha}^2(\mathbf{n})/4},$$

where the full phase is

$$i\Theta(\boldsymbol{R}) = i\kappa R - \frac{i\kappa}{4R} \left(3\overline{\alpha}(\mathbf{n}) - Tr\{\boldsymbol{A}^{-1}\} \right) - \frac{i\kappa^3}{8R} \left(\overline{\alpha^2}(\mathbf{n}) - \overline{\alpha}^2(\mathbf{n}) \right).$$

Exactly the same result may be obtained using the saddle-point method. However, further corrections are easier to calculate with the discussed approach.

Taking into account that

$$Tr\{\boldsymbol{A}^{-1}\} \sim \overline{\alpha}(\mathbf{n}) \sim \sigma_x^2 \sim \frac{1}{\sigma_p^2},$$

we obtain that the true expansion parameter is a combination of two different dimensionless ones:

$$\frac{\kappa}{R}\sigma_x^2 \sim \frac{\kappa}{R\sigma_p^2} \sim (\kappa\sigma_x)\frac{\sigma_x}{R} \ll 1.$$

For the neutrino oscillations problem, $\kappa = \sqrt{E_j^2 - m_j^2} \approx E_j$, so the parameter

$$(E_j\sigma_x)\frac{\sigma_x}{R} \ll 1, \quad R = |\boldsymbol{R}|,$$

defines the application conditions for the Grimus-Stockinger formula.

The 4-dimensional case.

The above technique may be applied to the 4-dimensional integral of the same problem:

$$J(R) = \int \frac{d^4q}{(2\pi)^2} \frac{e^{-i(qR)}\,\Phi(q)}{(q^2 - \widetilde{m}^2 + i0)}. \tag{1}$$

Assuming again, that $\Phi(q)$ admits 4-dimensional Fourier representation, for $\sqrt{R^2} = \sqrt{R^\mu R_\mu} \to \infty$, $\eta = R/\sqrt{R^2}$, $l = i\sqrt{R^2}$ one obtains:

$$J(R) \approx \frac{\widetilde{m}^2}{i} \sqrt{\frac{\pi}{2}} \frac{e^{-\widetilde{m}l}}{(\widetilde{m}l)^{3/2}} \left(1 + \frac{3}{8\widetilde{m}l}\right) \left\{1 - \frac{3(\eta\partial_q)}{2l} + \frac{\widetilde{m}}{2l}\left[(\eta\partial_q)^2 - \partial_q^2\right]\right\} \Phi(q)\Big|_{q=-\widetilde{m}\eta}, \tag{2}$$

representing the propagator in coordinate space:

$$\frac{1}{i} \int \frac{d^4q}{(2\pi)^4} \frac{e^{-i(qx)}}{(\widetilde{m}^2 - q^2 - i0)} = \frac{\widetilde{m}^2}{4\pi^2} h\left(\widetilde{m}^2(i0 - x^2)\right). \tag{3}$$

For a relativistic Gaussian wave packet with $\boldsymbol{A}^{-1} \sim \sigma_x^2$, this analoguesly gives:

$$J(R) \approx \frac{\widetilde{m}^2}{i}\sqrt{\frac{\pi}{2}}\frac{e^{-\widetilde{m}l}e^{-\widetilde{m}^2\overline{\alpha}(\eta)/4}}{(\widetilde{m}l)^{3/2}}\left(1+\frac{3}{8\widetilde{m}l}\right)$$
$$\left\{1-\frac{\widetilde{m}}{4l}\left[4\overline{\alpha}(\eta)-Tr\{\boldsymbol{A}^{-1}\}\right]-\frac{\widetilde{m}^3}{8l}\left[\overline{\alpha^2}(\eta)-(\overline{\alpha}(\eta))^2\right]\right\}, \quad (4)$$

where $\overline{\alpha}(\eta) = \left(\eta \boldsymbol{A}^{-1}\eta\right)$, $\overline{\alpha^2}(\eta) = \left(\eta \boldsymbol{A}^{-2}\eta\right)$.
The true parameter of expansion appears again as the product of the following two dimensionless parameters:

$$(\widetilde{m}\sigma_x)\frac{\sigma_x}{\sqrt{R^2}} \ll 1, \quad (5)$$

which implies, for example, that $(\widetilde{m}\sigma_x) \ll 1$.

Unfortunately, to obtain (2) and (4) we have used the well known expansion of the scalar propagator (3) in the integral (1):

$$J(R) = \frac{1}{i} \int d^4\rho \widetilde{m}^2 h \left(i0 - \widetilde{m}^2 (R-\rho)^2 \right) \phi(\rho), \tag{6}$$

with

$$\Phi(q) = \int d^4 r e^{i(qr)} \phi(r), \tag{7}$$

under the additional condition $\widetilde{m}|l| \gg 1$ giving the factor

$$\left(1 + \frac{3}{8\widetilde{m}l} \right), \tag{8}$$

which does not appear in the saddle-point approximation:

$$J(R) \approx \frac{\widetilde{m}^2}{i} \sqrt{\frac{\pi}{2}} \frac{e^{-\widetilde{m}l} e^{-\widetilde{m}^2 \overline{\alpha}(\eta)/4}}{(\widetilde{m}l)^{3/2}} \left\{ 1 - \frac{\widetilde{m}}{4l} \left[4\overline{\alpha}(\eta) - Tr\{\boldsymbol{A}^{-1}\} \right] - \frac{\widetilde{m}^3}{8l} \left[\overline{\alpha^2}(\eta) - \left(\overline{\alpha}(\eta) \right)^2 \right] \right\}, \tag{9}$$

under the same condition (5).

It's easy to show, that the conditions for three and four-dimensional cases are the same. Since

$$\mathbf{v} = \frac{\mathbf{p}}{E_p} = \frac{L}{T}, \tag{10}$$

then

$$R^2 = T^2 - L^2 = L^2\left(\frac{1}{\mathbf{v}^2} - 1\right) = \frac{L^2}{\mathbf{p}^2}\left(E_p^2 - \mathbf{p}^2\right) = \frac{L^2}{\mathbf{p}^2}\widetilde{m}^2, \tag{11}$$

so the condition (5):

$$(\widetilde{m}\sigma_x)\frac{\sigma_x}{\sqrt{R^2}} \ll 1, \tag{12}$$

may be rewritten as

$$(|\mathbf{p}|\sigma_x)\frac{\sigma_x}{L} \ll 1, \tag{13}$$

which is equivalent to the 3-dimensional condition for $E_p \approx |\mathbf{p}|$, $L = |\boldsymbol{R}|$.