Standard Model

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Free real scalar field

Lagrangian

$$L = \frac{1}{2} \left(\partial_\mu \varphi\right) \left(\partial^\mu \varphi\right) - \frac{m^2}{2} \varphi^2$$

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Functional integral $m^2 \to m^2 - i0$

$$<T\{\varphi(x_1)\varphi(x_2)\}> = \frac{\int e^{iS(\varphi)} \varphi(x_1)\varphi(x_2) D\varphi}{\int e^{iS(\varphi)} D\varphi}$$

$$S(\varphi) = \int L d^4 x \qquad D\varphi = \prod_x d\varphi(x)$$

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Generating functional

$$<T\{\varphi(x_1)\varphi(x_2)\}> = \left[\frac{1}{Z(J)}\frac{1}{i}\frac{\delta}{\delta J(x_1)}\frac{1}{i}\frac{\delta}{\delta J(x_2)}Z(J)\right]_{J=0}$$

$$Z(J) = \int e^{iS(\varphi,J)}D\varphi(x) \qquad S(\varphi, J) = \int (L + J\varphi)d^4x$$

Quadratic form

$$L = \frac{1}{2}\varphi \hat{M} \varphi \qquad \hat{M}(\partial) = -\partial^2 - m^2$$

Minimum of $S(\varphi, J)$

$$\hat{M}\varphi_0 + J = 0 \qquad \varphi_0 = -\hat{G}J \qquad \hat{G} = \hat{M}^{-1}$$

$$\varphi_0(x) = -\int G(x-y)J(y)d^4y \qquad \hat{M}G(x-y) = \delta(x-y)$$

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Momentum space

$$G(p) = M^{-1}(-ip) = \frac{1}{p^2 - m^2 + i0}$$

$$G(x) = \int G(p)e^{-ipx}\frac{d^4p}{(2\pi)^4}$$

Shift $\varphi = \varphi_0 + \varphi'$

$$S(\varphi, J) = \frac{1}{2} \int \left(-J\hat{G}J + \varphi' \hat{M} \varphi' \right) d^4x$$

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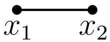
$$Z(J) = Z(0) \exp \left[-\frac{i}{2} \int J(x) G(x-y) J(y) \, d^4x \, d^4y \right]$$

Free propagator

$$<T\{\varphi(x_1)\varphi(x_2)\}> =$$



$$= iG(x_1 - x_2)$$

Free propagator

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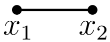
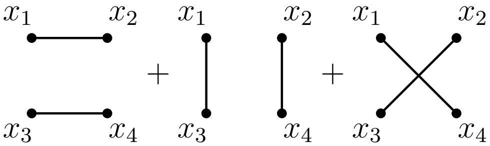


$$= iG(x_1 - x_2)$$

Wick theorem

$$\begin{aligned} &< T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\} > \\ &\quad = < T\{\varphi(x_1)\varphi(x_2)\} > < T\{\varphi(x_3)\varphi(x_4)\} > \\ &\quad + < T\{\varphi(x_1)\varphi(x_3)\} > < T\{\varphi(x_2)\varphi(x_4)\} > \\ &\quad + < T\{\varphi(x_1)\varphi(x_4)\} > < T\{\varphi(x_2)\varphi(x_3)\} > \end{aligned}$$

$=$



Interaction

$$L = L_0 + L_1 \qquad L_0 = \frac{1}{2}\left(\partial_\mu \varphi\right)\left(\partial^\mu \varphi\right) - \frac{m^2}{2}\varphi^2 \qquad L_1 = -\frac{\lambda}{4!}\varphi^4$$

Symmetry $\varphi \rightarrow -\varphi$

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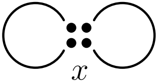
Symmetry $\varphi \to -\varphi$

Perturbation theory

$$Z(J) = \int e^{iS_0(\varphi, J)}\left(1 + iS_1(\varphi) + \cdots\right) D\varphi$$

Vacuum $\to$ vacuum

$$<0|0> = <1>_0 - \frac{i}{4!}\lambda \int <\varphi^4(x)>_0 \, d^4x + \cdots$$

$$= 1 - \frac{i}{4!}\lambda$$ 

$$+ \cdots$$

Vacuum energy density $\to$ phase

Vacuum → vacuum

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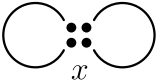
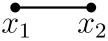
$$= 1 - \frac{i}{4!}\lambda$$ 

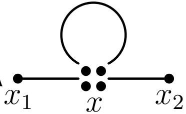
$$+ \cdots$$

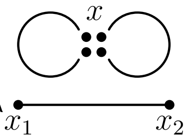
Vacuum energy density → phase

Propagator

$$\int e^{iS_0(\varphi)}(1 + iS_1(\varphi) + \cdots)\varphi(x_1)\varphi(x_2)D\varphi =$$



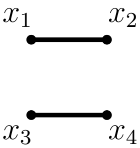
$$- \frac{i}{4!}\lambda$$ 

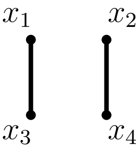
$$- \frac{i}{4!}\lambda$$ 

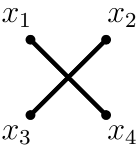
$$+ \cdots$$

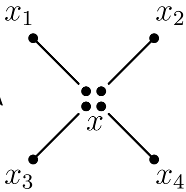
Diagrams with vacuum bubbles cancel

$$<T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\}>$$

$=$ 

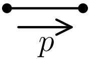
$+$ 

$+$ 

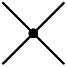
$-\dfrac{i}{4!}\lambda$ 

$+\cdots$

Feynman rules



$= iG_0(p) \qquad G_0(p) = \dfrac{1}{p^2 - m^2 + i0}$



$= -i\lambda$

Beware of symmetry factors!

Renormalization

Lagrangian

$$L = \frac{1}{2}\left(\partial_\mu \varphi_0\right)\left(\partial^\mu \varphi_0\right) - \frac{m_0^2}{2}\varphi_0^2 - \frac{\lambda_0}{4!}\varphi_0^4$$

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Renormalization

$$\varphi_0 = Z_\varphi^{1/2}\varphi \qquad m_0 = Z_m m \qquad \lambda_0 = Z_\lambda \lambda$$

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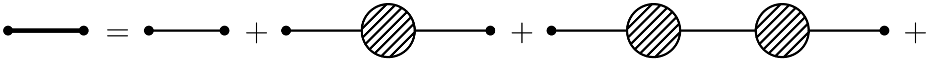
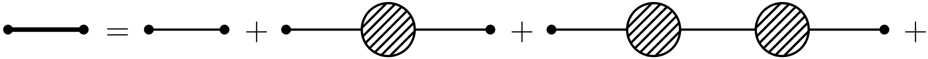
In $d = 4 - 2\varepsilon$ dimensions: $[L] = d$, $[\varphi] = 1 - \varepsilon$, $[\lambda] = 2\varepsilon$

Propagator



$$iG = iG_0 + iG_0(-i)\Sigma iG_0 + iG_0(-i)\Sigma iG_0(-i)\Sigma iG_0 + \cdots$$

Propagator



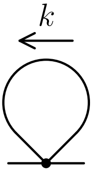
$$iG = iG_0 + iG_0(-i)\Sigma iG_0 + iG_0(-i)\Sigma iG_0(-i)\Sigma iG_0 + \cdots$$

$$G(p) = G_0(p) + G_0(p)\Sigma(p)G(p) \qquad G^{-1}(p) = G_0^{-1}(p) - \Sigma(p)$$

$$G(p) = \frac{1}{p^2 - m_0^2 - \Sigma(p) + i0}$$

1 loop

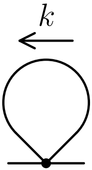
$$-i\Sigma(p) =$$



$$\Sigma(p) = -i\frac{\lambda_0}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{m_0^2 - k^2 - i0}$$

Symmetry factor $\frac{1}{2}$

1 loop

$-i\Sigma(p) =$ 

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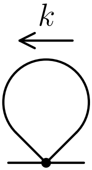
Symmetry factor $\frac{1}{2}$

$$\int \frac{d^d k}{(m^2 - k^2 - i0)^n} = i\pi^{d/2} m^{d-2n} V(n) \qquad V(n) = \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)}$$

1 loop

$-i\Sigma(p) =$ 

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$$\Sigma(p) = \frac{1}{2}\frac{\lambda_0 m_0^{d-2}}{(4\pi)^{d/2}}\Gamma\left(1 - \frac{d}{2}\right)$$

On-shell renormalization: mass

$$G(p) = \frac{1}{p^2 - m_0^2 - \Sigma(p^2) - i0}$$

has pole at $p^2 = m_{\text{os}}^2$

$$m_{\text{os}}^2 - m_0^2 - \Sigma(m_{\text{os}}^2) = 0$$

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1 loop

$$m_0^2 = m_{\text{os}}^2 \left[1 + \frac{1}{d-2} \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) + \cdots \right]$$

$$Z_m^{\text{os}} = 1 + \frac{1}{2(d-2)} \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) + \cdots$$

On-shell renormalization: field

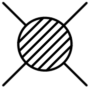
At $p^2 \to m_{\text{os}}^2$

$$G_{\text{os}}(p) \to G_0(p) = \frac{1}{p^2 - m_{\text{os}}^2 + i0}$$

1 loop

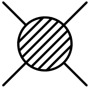
$$Z_\varphi^{\text{os}} = 1 + \mathcal{O}(\lambda^2)$$

Vertex

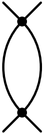
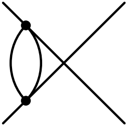


$$= -i\lambda_0\Gamma(p_i) \qquad \Gamma(p_i) = 1 + \Lambda(p_i)$$

Vertex

 $= -i\lambda_0 \Gamma(p_i) \qquad \Gamma(p_i) = 1 + \Lambda(p_i)$

1 loop

$\Lambda(p_i) =$  $+$  $+$ 

$$= -\frac{1}{2} \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \left[f\left(\frac{s}{m_0^2}\right) + f\left(\frac{t}{m_0^2}\right) + f\left(\frac{u}{m_0^2}\right) \right]$$

where

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{[m^2 - k^2 - i0]\,[m^2 - (k+p)^2 - i0]} = \frac{i m^{-2\varepsilon}}{(4\pi)^{d/2}} f\left(\frac{p^2}{m^2}\right)$$

1 loop

$$f(0) = \Gamma(\varepsilon)$$

$f(x) - f(0)$ finite at $\varepsilon \to 0$

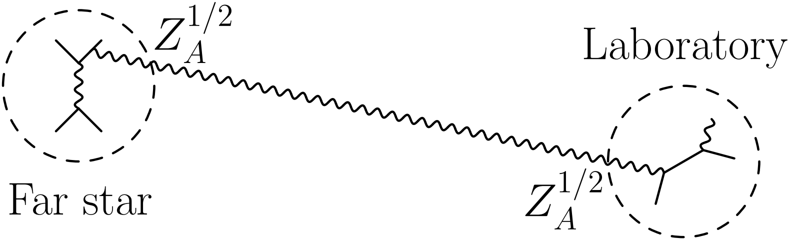
$$\Gamma = 1 - \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \left(\frac{3}{2} + \mathcal{O}(\varepsilon) \right)$$

$\overline{\text{MS}}$ renormalization

$$\frac{\lambda_0}{(4\pi)^{d/2}} = \mu^{2\varepsilon} \frac{\lambda(\mu)}{(4\pi)^2} Z_\lambda(\lambda(\mu)) e^{\gamma\varepsilon}$$

$$Z_\lambda(\lambda) = 1 + \frac{z_1}{\varepsilon} \frac{\lambda}{(4\pi)^2} + \left(\frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon} \right) \frac{\lambda^2}{(4\pi)^4} + \cdots$$

LSZ reduction formula



Free propagator $\Rightarrow$ spin wave functions
Full propagator (very) close to the mass shell

S-matrix element = vertex $\times$ $(Z_i^{\rm os})^{1/2}$ for each i

Renormalization of the coupling constant

Vertex

$$\Gamma = Z_\Gamma \Gamma_r$$

Γ_r finite at $\varepsilon \to 0$

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Γ_r finite at $\varepsilon \to 0$

Scattering amplitude

$$\lambda_0 \Gamma Z_\varphi^2 = Z_\lambda Z_\Gamma Z_\varphi^2 \lambda \Gamma_r$$

must be finite at $\varepsilon \to 0$ ($Z_\varphi^{\text{os}} / Z_\varphi = \text{finite}$)

Renormalization of the coupling constant

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$$Z_\lambda Z_\Gamma Z_\varphi^2 = \text{finite}$$

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$$Z_\lambda Z_\Gamma Z_\varphi^2 = 1$$

$$Z_\lambda = \left(Z_\Gamma Z_\varphi^2\right)^{-1}$$

Renormalization group

$$Z_\lambda = 1 + z_1 \frac{\lambda}{(4\pi)^2 \varepsilon} + \cdots = 1 + \frac{3}{2} \frac{\lambda}{(4\pi)^2 \varepsilon} + \cdots$$

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λ_0 does not depend on μ

$$\frac{d \log \lambda}{d \log \mu} = -2\varepsilon + \frac{d \log Z_\lambda}{d \log \lambda} \frac{d \log \lambda}{d \log \mu} = -2\varepsilon + 2 z_1 \frac{\lambda}{(4\pi)^2} + \cdots$$

Renormalization group

$$Z_\lambda = 1 + z_1 \frac{\lambda}{(4\pi)^2 \varepsilon} + \cdots = 1 + \frac{3}{2} \frac{\lambda}{(4\pi)^2 \varepsilon} + \cdots$$

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Renormalization group equation at $\varepsilon = 0$

$$\frac{d}{d \log \mu} \frac{\lambda(\mu)}{(4\pi)^2} = \beta(\lambda(\mu))$$

$$\beta(\lambda) = \beta_0 \frac{\lambda^2}{(4\pi)^4} + \cdots$$

$$\beta_0 = 2 z_1 = 3$$

$\lambda(\mu)$ grows with μ

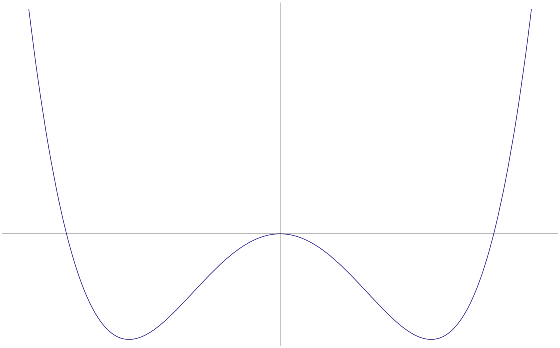
Spontaneous symmetry breaking

$$m^2 \to -\mu^2$$

$$L = \frac{1}{2}\left(\partial_\mu \varphi\right)\left(\partial^\mu \varphi\right) - V(\varphi)$$

$$V(\varphi) = -\frac{\mu^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4 = \frac{\lambda}{4!}(\varphi^2 - v^2)^2 + \text{const} \qquad v^2 = 3!\frac{\mu^2}{\lambda}$$

Out[5]=



Quantum mechanics

Ground state is even, splitting $\sim$ tunneling amplitude

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Quantum field theory

$N \to \infty$ large space regions $\gg$ correlation length

Tunneling amplitude

$$A^N \to 0$$

2 degenerate vacua, each breaks the $\varphi \to -\varphi$ symmetry

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Quantum field theory

$N \to \infty$ large space regions $\gg$ correlation length

Tunneling amplitude

$$A^N \to 0$$

2 degenerate vacua, each breaks the $\varphi \to -\varphi$ symmetry

Suppose we are in the $<\varphi> = +v$ vacuum

$$\varphi = v + h$$

$$L = \frac{1}{2}\left(\partial_\mu h\right)\left(\partial^\mu h\right) - \mu^2 h^2 - \frac{\lambda v}{3!}h^3 - \frac{\lambda}{4!}h^4$$

$m = \sqrt{2}\mu$, triple and quadruple vertices

Renormalization

$$\varphi_0 = v_0 + h_0 \qquad (<h_0> = 0)$$

$$L = \frac{1}{2}\left(\partial_\mu h_0\right)\left(\partial^\mu h_0\right) - \frac{m_0^2}{2} h_0^2$$

$$+ \left(\mu_0^2 - \frac{\lambda_0}{3!} v_0^2\right) v_0 h_0 - \frac{\lambda_0 v_0}{3!} h_0^3 - \frac{\lambda_0}{4!} h_0^4$$

$$m_0^2 = \frac{\lambda_0}{2} v_0^2 - \mu_0^2$$

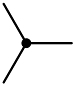
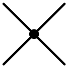
Renormalization

$$\varphi_0 = v_0 + h_0 \qquad (<h_0> = 0)$$

$$L = \frac{1}{2}\left(\partial_\mu h_0\right)\left(\partial^\mu h_0\right) - \frac{m_0^2}{2} h_0^2 + \left(\mu_0^2 - \frac{\lambda_0}{3!} v_0^2\right) v_0 h_0 - \frac{\lambda_0 v_0}{3!} h_0^3 - \frac{\lambda_0}{4!} h_0^4$$

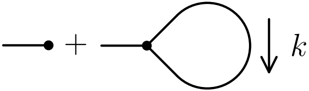
$$m_0^2 = \frac{\lambda_0}{2} v_0^2 - \mu_0^2$$

 $= i\left(\mu_0^2 - \frac{\lambda_0}{3!} v_0^2\right) v_0$

 $= -i\lambda_0 v_0$  $= -i\lambda_0$

Vacuum expectation value

$$\langle h_0 \rangle = 0 =$$

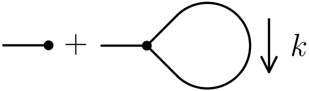


$$= i\left(\mu_0^2 - \frac{\lambda_0}{3!}v_0^2\right)v_0 - \frac{\lambda_0 v_0}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{m_0^2 - k^2 - i0}$$

$$\frac{\lambda_0}{3!}v_0^2 - \mu_0^2 = \frac{m_0^2}{d-2}\frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)$$

Vacuum expectation value

$$<h_0> = 0 =$$



$$= i \left(\mu_0^2 - \frac{\lambda_0}{3!} v_0^2 \right) v_0 - \frac{\lambda_0 v_0}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{m_0^2 - k^2 - i0}$$

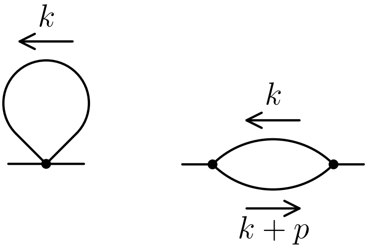
$$\frac{\lambda_0}{3!} v_0^2 - \mu_0^2 = \frac{m_0^2}{d-2} \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon)$$

In the correction

$$v_0^2 = 3! \frac{\mu_0^2}{\lambda_0} \qquad m_0^2 = 2\mu_0^2$$

$$v_0^2 = 3! \frac{\mu_0^2}{\lambda_0} \left[1 + \frac{2}{d-2} \frac{\lambda_0 (2\mu_0^2)^{-\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) + \cdots \right]$$

Self energy



$$\Sigma(p) = -\frac{\lambda_0 m_0^{d-2}}{(4\pi)^{d/2}}\left[\frac{\Gamma(\varepsilon)}{d-2} + \frac{3}{2} f\left(\frac{p^2}{m_0^2}\right)\right]$$

On-shell mass and field renormalization

Mass

$$m_0^2 = m_{\rm os}^2 \left[1 + \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \left(\frac{\Gamma(\varepsilon)}{d-2} + \frac{3}{2} f(1) \right) + \cdots \right]$$

$$= m_{\rm os}^2 \left[1 + \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \left(2 + \frac{\varepsilon}{2} + \frac{3}{2} \left(f(1) - f(0) \right) \varepsilon + \mathcal{O}(\varepsilon^2) \right) + \cdot$$

$\left(f(1) - f(0) \right)_{\varepsilon=0}$ is a number

On-shell mass and field renormalization

Mass

$$m_0^2 = m_{\text{os}}^2 \left[1 + \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \left(\frac{\Gamma(\varepsilon)}{d-2} + \frac{3}{2} f(1) \right) + \cdots \right]$$

$$= m_{\text{os}}^2 \left[1 + \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \left(2 + \frac{\varepsilon}{2} + \frac{3}{2} (f(1) - f(0)) \varepsilon + \mathcal{O}(\varepsilon^2) \right) + \cdot$$

$(f(1) - f(0))_{\varepsilon=0}$ is a number

Field $p^2 \to m_{\text{os}}^2$

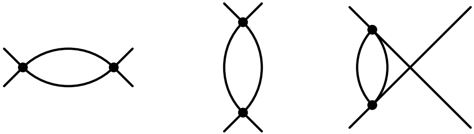
$$G(p) = \frac{1}{p^2 - m_0^2 - \Sigma(p^2) + i0} = \frac{Z_h^{\text{os}}}{p^2 - m_{\text{os}}^2 + i0}$$

$$Z_h^{\text{os}} = \frac{1}{1 - \left(\frac{d\Sigma(p^2)}{dp^2} \right)_{p^2 = m_{\text{os}}^2}} = 1 - \frac{3}{2} \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} f'(1) + \cdots$$

$f'(1)$ is finite at $\varepsilon \to 0$

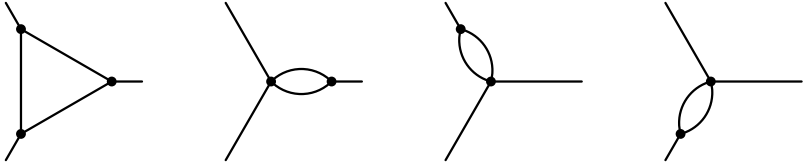
4-leg vertex

UV divergences only in



$\overline{\text{MS}}$ renormalization of λ –
the same as in the unbroken theory

3-leg vertex



First – convergent

$$\Gamma = 1 - \frac{1}{2}\frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}}\left[f\left(\frac{p_1^2}{m_0^2}\right) + f\left(\frac{p_2^2}{m_0^2}\right) + f\left(\frac{p_3^2}{m_0^2}\right) + \text{finite}\right]$$

$$= 1 - \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\left(\frac{3}{2} + \mathcal{O}(\varepsilon)\right)$$

Gives the same renormalization of λ

Complex scalar field

$$L = \left(\partial_\mu \varphi^*\right)\left(\partial^\mu \varphi\right) - m^2 \varphi^* \varphi - \frac{\lambda}{4}\left(\varphi^* \varphi\right)^2$$

$U(1)$ symmetry $\varphi \to e^{i\alpha}\varphi$

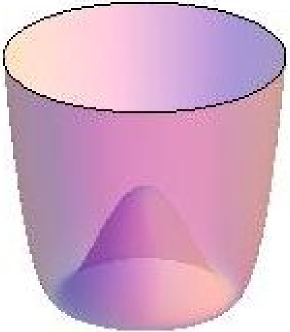
Spontaneous symmetry breaking

$m^2 \to -\mu^2$

$$L = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - V(\varphi)$$

$$V(\varphi) = -\mu^2 \varphi^* \varphi + \frac{\lambda}{4} (\varphi^* \varphi)^2 = \frac{\lambda}{4} \left(\varphi^* \varphi - \frac{v^2}{2} \right)^2 + \text{const}$$

$$v^2 = 4 \frac{\mu^2}{\lambda}$$



Vacua

$$<\theta|\varphi|\theta> = \frac{v}{\sqrt{2}} e^{i\theta}$$

Suppose we are in the $\theta = 0$ vacuum

$$\varphi = \frac{1}{\sqrt{2}}(v + h + ib)$$

$$L = \frac{1}{2}(\partial_\mu h)(\partial^\mu h) + \frac{1}{2}(\partial_\mu b)(\partial^\mu b) - \mu^2 h^2 - \frac{\lambda v}{4} h(h^2 + b^2) - \frac{\lambda}{16}(h^2 + b^2)^2$$

$$v = \frac{2\mu}{\sqrt{\lambda}}$$

h – $m = \sqrt{2}\mu$

b – massless Goldstone boson

Polar coordinates

$$\varphi = \frac{1}{\sqrt{2}}(v+h)e^{i\theta/v}$$

$$L = \frac{1}{2}\left(\partial_\mu h\right)\left(\partial^\mu h\right) + \frac{1}{2}\left(1+\frac{h}{v}\right)^2\left(\partial_\mu \theta\right)\left(\partial^\mu \theta\right) - \mu^2 h^2 - \frac{\lambda v}{4}h^3 - \frac{\lambda}{16}h^4$$

h – $m = \sqrt{2}\mu$

θ – massless Goldstone boson

Polar coordinates

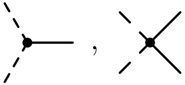
$$\varphi = \frac{1}{\sqrt{2}}(v+h)e^{i\theta/v}$$

$$L = \frac{1}{2}\left(\partial_\mu h\right)\left(\partial^\mu h\right) + \frac{1}{2}\left(1+\frac{h}{v}\right)^2\left(\partial_\mu \theta\right)\left(\partial^\mu \theta\right) - \mu^2 h^2$$

$$- \frac{\lambda v}{4}h^3 - \frac{\lambda}{16}h^4$$

h – $m = \sqrt{2}\mu$

θ – massless Goldstone boson



$\sim p_1 \cdot p_2$

$$\Sigma_\theta(p \to 0) \to 0$$

θ remains massless

Jacobian

$$\prod_x (v + h(x)) \sim \int e^{iS_c} D\bar{c}\, Dc$$

$$S_c = \int L_c d^4x \qquad L_c = -m_c^2 \bar{c} \left(1 + \frac{h}{v}\right) c$$

c – scalar fermion field (ghost), propagator

$$G_c(p) = -\frac{1}{m_c^2}$$

Jacobian

$$\prod_x (v + h(x)) \sim \int e^{iS_c} D\bar{c}\, Dc$$

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c – scalar fermion field (ghost), propagator

$$G_c(p) = -\frac{1}{m_c^2}$$

Anticommuting variables (always dimensionless)

$$c_i c_j + c_j c_i = 0 \qquad c_i^2 = 0$$

$$\int dc = 0 \qquad \int c\, dc = 1$$

$$e^{-a\bar{c}c} = 1 - a\bar{c}c \qquad \int e^{-a\bar{c}c}\, d\bar{c}\, dc = a$$

Goldstone theorem

N real fields

$$L = \frac{1}{2}\left(\partial_\mu \varphi_i\right)\left(\partial^\mu \varphi_i\right) - V(\varphi) \qquad V(\varphi) = \frac{m^2}{2}\varphi_i\varphi_i + \frac{\lambda}{4!}\left(\varphi_i\varphi_i\right)^2$$

$SO(N)$ symmetry, $N(N-1)/2$ generators
(rotations in coordinate planes)

Goldstone theorem

N real fields

$$L = \frac{1}{2}\left(\partial_\mu \varphi_i\right)\left(\partial^\mu \varphi_i\right) - V(\varphi) \qquad V(\varphi) = \frac{m^2}{2}\varphi_i\varphi_i + \frac{\lambda}{4!}\left(\varphi_i\varphi_i\right)^2$$

$SO(N)$ symmetry, $N(N-1)/2$ generators
(rotations in coordinate planes)
$m^2 \to -\mu^2$

$$V(\varphi) = \frac{\lambda}{4!}\left(\varphi_i\varphi_i - v^2\right)^2$$

Suppose we are in the vacuum $<\varphi_i> = v\delta_{iN}$
$SO(N-1)$ symmetry (with $(N-1)(N-2)/2$ generators) remains;
$N-1$ generators are broken
$N-1$ flat directions $\Rightarrow$ massless Goldstone bosons

Scalar electrodynamics

$$L = (\partial_\mu \varphi^*) (\partial^\mu \varphi) - m^2 \varphi^* \varphi - \frac{\lambda}{4} (\varphi^* \varphi)^2$$

Symmetry $\varphi \to e^{i\alpha}\varphi$, $\alpha = \text{const.}$ $\alpha(x)$?

Scalar electrodynamics

$$L = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi - \frac{\lambda}{4} (\varphi^* \varphi)^2$$

Symmetry $\varphi \to e^{i\alpha}\varphi$, $\alpha = \text{const.}$ $\alpha(x)$?

$$D_\mu \varphi = (\partial_\mu - ieA_\mu)\varphi \qquad D_\mu \varphi \to e^{i\alpha} D_\mu \varphi$$

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha$$

Scalar electrodynamics

$$L = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi - \frac{\lambda}{4} (\varphi^* \varphi)^2$$

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$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha$$

$$[D_\mu, D_\nu]\varphi \to e^{i\alpha}[D_\mu, D_\nu]\varphi$$

$$[D_\mu, D_\nu]\varphi = -ieF_{\mu\nu}\varphi \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Scalar electrodynamics

$$L = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi - \frac{\lambda}{4}(\varphi^* \varphi)^2$$

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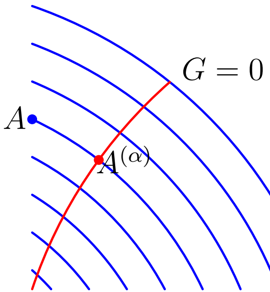
$$[D_\mu, D_\nu]\varphi = -ieF_{\mu\nu}\varphi \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$L = (D_\mu \varphi)^* (D^\mu \varphi) - m^2 \varphi^* \varphi - \frac{\lambda}{4}(\varphi^* \varphi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

invariand with respect to

$$\varphi(x) \to e^{i\alpha(x)}\varphi(x) \qquad A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$$

Gauge fields



Orbits of the gauge group $A \to A^{(\alpha)}$
Gauge $G(A^{(\alpha)}(x)) = 0$ — unique solution $\alpha(x)$
for any given $A(x)$
The surface $G = 0$ intersects any orbit at 1 point

Faddeev–Popov determinant

$$\Delta^{-1}(A) = \int \prod_x \delta(G(A^{(\alpha)}(x)))\, D\alpha$$

Near the surface $G(A^{(\alpha_0)}) = 0$:

$$\delta G(A(x)) = \hat{M}\delta\alpha(x)$$

$$\Delta^{-1}(A) = \int \delta(\hat{M}\alpha(x))\, D\alpha = 1/\det \hat{M}$$

Faddeev–Popov determinant

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$$\delta G(A(x)) = \hat{M}\delta\alpha(x)$$

$$\Delta^{-1}(A) = \int \delta(\hat{M}\alpha(x))\, D\alpha = 1/\det \hat{M}$$

$$Z(J) = \int e^{iS(A)}\, DA$$

$$= \int \Delta(A) \prod_x \delta(G(A^{(\alpha)}(x)))\, e^{iS(A)}\, D\alpha\, DA$$

$$= \left(\prod_x \int d\alpha\right) \times \int \Delta(A) \prod_x \delta(G(A(x)))\, e^{iS(A)}\, DA$$

Ghosts

$$\Delta(A) = \det \hat{M} = \int e^{iS_c}\, D\bar{c}\, Dc$$

$$S_c = \int L_c d^4x \qquad L_c \sim \bar{c}\hat{M}c$$

Ghosts

$$\Delta(A) = \det \hat{M} = \int e^{iS_c}\, D\bar{c}\, Dc$$

$$S_c = \int L_c d^4x \qquad L_c \sim \bar{c}\hat{M}c$$

Generalized Lorenz gauge $G(A(x)) = \partial_\mu A^\mu(x) - \omega(x)$

$$\delta G(x) = \partial_\mu \delta A^\mu(x) = \frac{1}{e}\partial_\mu \partial^\mu \delta\alpha(x)$$

$$\hat{M} = \frac{1}{e}\partial_\mu \partial^\mu \qquad \det \hat{M} = \text{const}$$

$$L_c = -\bar{c}\partial_\mu \partial^\mu c \Rightarrow (\partial_\mu \bar{c})\,(\partial^\mu c)$$

Covariant gauge

$$Z(J) = \int \prod_x \delta(\partial_\mu A^\mu(x) - \omega(x))\, e^{iS}\, DA$$

Covariant gauge

$$Z(J) = \int \prod_x \delta(\partial_\mu A^\mu(x) - \omega(x))\, e^{iS}\, DA$$

$$\times\, e^{-\frac{i}{2\xi}\int \omega^2(x)\, d^4x}\, D\omega$$

Covariant gauge

$$Z(J) = \int \prod_x \delta(\partial_\mu A^\mu(x) - \omega(x))\, e^{iS}\, DA$$

$$\times\, e^{-\frac{i}{2\xi}\int \omega^2(x)\, d^4x}\, D\omega$$

$$= \int e^{i(S+S_\xi)}\, DA \qquad L_\xi = -\frac{1}{2\xi}\,(\partial_\mu A^\mu)^2$$

Photon propagator

$$L = -\frac{1}{2}\left(\partial_\mu A_\nu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)\left(\partial_\nu A^\nu\right) \rightarrow \frac{1}{2}A^\mu M_{\mu\nu} A^\nu$$

$$M_{\mu\nu}(\partial) = g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + \frac{1}{\xi}\partial_\mu\partial_\nu$$

$$M_{\mu\nu}(-ip) = -p^2\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} + \frac{1}{\xi}\frac{p_\mu p_\nu}{p^2}\right]$$

$$M^{-1}_{\mu\nu}(-ip) = -\frac{1}{p^2}\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} + \xi\frac{p_\mu p_\nu}{p^2}\right]$$

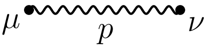
Photon propagator

$$L = -\frac{1}{2}\left(\partial_\mu A_\nu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)\left(\partial_\nu A^\nu\right) \to \frac{1}{2} A^\mu M_{\mu\nu} A^\nu$$

$$M_{\mu\nu}(\partial) = g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + \frac{1}{\xi}\partial_\mu\partial_\nu$$

$$M_{\mu\nu}(-ip) = -p^2\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} + \frac{1}{\xi}\frac{p_\mu p_\nu}{p^2}\right]$$

$$M^{-1}_{\mu\nu}(-ip) = -\frac{1}{p^2}\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} + \xi\frac{p_\mu p_\nu}{p^2}\right]$$



$$= -iD^0_{\mu\nu}(p)$$

$$D^0_{\mu\nu}(p) = \frac{1}{p^2}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right]$$

Abelian Higgs model

$m^2 \to -\mu^2$

$$L = (D_\mu \varphi)^* (D^\mu \varphi) + \mu^2 \varphi^* \varphi - \frac{\lambda}{4} (\varphi^* \varphi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\varphi = \frac{1}{\sqrt{2}} (v + h) e^{i\theta/v} \qquad v^2 = \frac{4\mu^2}{\lambda}$$

Abelian Higgs model

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$$L = (D_\mu \varphi)^* (D^\mu \varphi) + \mu^2 \varphi^* \varphi - \frac{\lambda}{4} (\varphi^* \varphi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$\varphi = \frac{1}{\sqrt{2}} (v + h) e^{i\theta/v} \qquad v^2 = \frac{4\mu^2}{\lambda}$$

Unitary gauge $\theta = 0$

$$D_\mu \varphi = \frac{1}{\sqrt{2}} \left[\partial_\mu h - ie(v + h) A_\mu \right]$$

$$L = \frac{1}{2} (\partial_\mu h) (\partial^\mu h) + \frac{e^2}{2} (v + h)^2 A_\mu A^\mu - \mu^2 h^2 - \frac{\lambda v}{4} h^3 - \frac{\lambda}{16} h^4 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

Photon

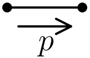
$$L_A = \frac{1}{2} A^\mu M_{\mu\nu} A^\nu$$

$$M_{\mu\nu}(\partial) = g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + m_A^2 g_{\mu\nu} \qquad m_A = ev$$

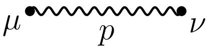
$$M_{\mu\nu}(-ip) = -\left(p^2 - m_A^2\right)\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + m_A^2 \frac{p_\mu p_\nu}{p^2}$$

$$M_{\mu\nu}^{-1}(-ip) = -\frac{1}{p^2 - m_A^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) + \frac{1}{m_A^2}\frac{p_\mu p_\nu}{p^2}$$

Unitary gauge: propagators

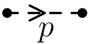


$$= iG(p) \qquad G(p) = \frac{1}{p^2 - m_h^2 + i0}$$



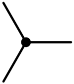
$$= -iD_{\mu\nu}(p)$$

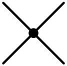
$$D_{\mu\nu}(p) = \frac{1}{p^2 - m_A^2}\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{m_A^2}\right]$$

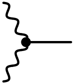


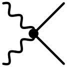
$$= iG_c(p) \qquad G_c(p) = -\frac{1}{m_c^2}$$

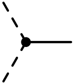
Unitary gauge: vertices

 $= -\frac{3}{2} i\lambda v$

 $= -\frac{3}{2} i\lambda$

 $= 2ie^2 v g_{\mu\nu}$

 $= 2ie^2 g_{\mu\nu}$

 $= -i\frac{m_c^2}{v}$

Renormalizable gauge

$$\varphi = \frac{1}{\sqrt{2}}(v + h + ib)$$

$$D_\mu \varphi = \frac{1}{\sqrt{2}}\left[\partial_\mu h + ebA_\mu + i\partial_\mu b - ie(v+h)A_\mu\right]$$

$$V(\varphi) = \mu^2 h^2 + \frac{\lambda v}{2} h(h^2 + b^2) + \frac{\lambda}{16}(h^2 + b^2)^2$$

$$L = \frac{1}{2}\left(\partial_\mu h\right)\left(\partial^\mu h\right) - \mu^2 h^2 + \frac{1}{2}\left(\partial_\mu b\right)\left(\partial^\mu b\right) - evA^\mu \partial_\mu b$$
$$- \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{e^2 v^2}{2} A_\mu A^\mu$$
$$- \frac{\lambda v}{2} h(h^2 + b^2) - \frac{\lambda}{16}(h^2 + b^2)^2 + eA^\mu\left(b\partial_\mu h - h\partial_\mu b\right)$$
$$+ e^2 v h A_\mu A^\mu + \frac{e^2}{2}(h^2 + b^2) A_\mu A^\mu$$

R_ξ gauge

Gauge $G = 0$

$$L_\xi = -\frac{1}{2\xi} G^2 \qquad L_c \sim \bar{c} \frac{\delta G}{\delta \alpha} c$$

R_ξ gauge

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$$L_\xi = -\frac{1}{2\xi}G^2 \qquad L_c \sim \bar{c}\frac{\delta G}{\delta \alpha}c$$

$$G = \partial_\mu A^\mu + \xi e v b$$

$$L_\xi = -\frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 - evb\partial_\mu A^\mu - \frac{\xi e^2 v^2}{2}b^2$$

R_ξ gauge

Gauge $G = 0$

$$L_\xi = -\frac{1}{2\xi} G^2 \qquad L_c \sim \bar{c} \frac{\delta G}{\delta \alpha} c$$

$$G = \partial_\mu A^\mu + \xi e v b$$

$$L_\xi = -\frac{1}{2\xi} \left(\partial_\mu A^\mu \right)^2 - e v b \partial_\mu A^\mu - \frac{\xi e^2 v^2}{2} b^2$$

$$\delta A_\mu = \frac{1}{e} \partial_\mu \delta\alpha \qquad \delta\varphi = i \delta\alpha \varphi \qquad \delta b = (v + h) \delta\alpha$$

$$\delta G = \left[\frac{1}{e} \partial^2 + \xi e v (v + h) \right] \delta\alpha$$

$$L_c = \left(\partial_\mu \bar{c} \right) \left(\partial^\mu c \right) - \xi e^2 v (v + h) \bar{c} c$$

Full Lagrangian

$$L + L_\xi + L_c = \frac{1}{2}\left(\partial_\mu h\right)\left(\partial^\mu h\right) - \frac{m_h^2}{2}h^2$$
$$+ \frac{1}{2}\left(\partial_\mu b\right)\left(\partial^\mu b\right) - \frac{\xi m_A^2}{2}b^2$$
$$+ \left(\partial_\mu \bar{c}\right)\left(\partial^\mu c\right) - \xi m_A^2 \bar{c}c$$
$$- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)^2 + \frac{m_A^2}{2}A_\mu A^\mu$$
$$- \frac{\lambda v}{2}h(h^2 + b^2) - \frac{\lambda}{16}(h^2 + b^2)^2$$
$$+ eA^\mu\left(b\partial_\mu h - h\partial_\mu b\right) + e^2 v h A_\mu A^\mu + \frac{e^2}{2}(h^2 + b^2)A_\mu A^\mu$$
$$- \xi e^2 v h \bar{c}c$$

$$m_h = \sqrt{2}\mu \qquad m_A = ev \qquad v = \frac{2\mu}{\sqrt{\lambda}}$$

Photon

$$L = \frac{1}{2} A^\mu M_{\mu\nu} A^\nu$$

$$M_{\mu\nu}(\partial) = g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + \frac{1}{\xi}\partial_\mu\partial_\nu + m_A^2 g_{\mu\nu}$$

$$M_{\mu\nu}(-ip) = -\left(p^2 - m_A^2\right)\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) - \left(\frac{p^2}{\xi} - m_A^2\right)\frac{p_\mu p_\nu}{p^2}$$

$$M_{\mu\nu}^{-1}(-ip) = -\frac{1}{p^2 - m_A^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right) - \frac{\xi}{p^2 - \xi m_A^2}\frac{p_\mu p_\nu}{p^2}$$

$$D_{\mu\nu}(p) = \frac{1}{p^2 - m_A^2}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2 - \xi m_A^2}\right]$$

Particular cases

't Hooft–Landau gauge $\xi \to 0$

$$G_b(p) = G_c(p) = \frac{1}{p^2} \qquad D_{\mu\nu}(p) = \frac{1}{p^2 - m_A^2}\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right]$$

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't Hooft–Feynman gauge $\xi = 1$

$$G_b(p) = G_c(p) = \frac{1}{p^2 - m_A^2} \qquad D_{\mu\nu}(p) = \frac{g_{\mu\nu}}{p^2 - m_A^2}$$

Particular cases

't Hooft–Landau gauge $\xi \to 0$

$$G_b(p) = G_c(p) = \frac{1}{p^2} \qquad D_{\mu\nu}(p) = \frac{1}{p^2 - m_A^2}\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right]$$

't Hooft–Feynman gauge $\xi = 1$

$$G_b(p) = G_c(p) = \frac{1}{p^2 - m_A^2} \qquad D_{\mu\nu}(p) = \frac{g_{\mu\nu}}{p^2 - m_A^2}$$

Unitary gauge $\xi \to \infty$

$$D_{\mu\nu}(p) = \frac{1}{p^2 - m_A^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{m_A^2}\right)$$

b disappears; $c \to c/\sqrt{\xi}$

$$L_c = -m_A^2 \bar{c}c - e^2 v h \bar{c}c$$

Nonabelian gauge fields

$$L = (\partial_\mu \varphi)^+ (\partial^\mu \varphi)$$

Invariant with respect to $SU(N)$: $\varphi(x) \to U\varphi(x)$

Infinitesimal tramsformation

$$U = 1 + i\alpha^a T^a$$

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Infinitesimal tramsformation

$$U = 1 + i\alpha^a T^a$$

How to make it invariant for $\varphi(x) \to U(x)\varphi(x)$?

$$\partial_\mu \varphi \;\Rightarrow\; D_\mu \varphi$$

$$D_\mu \varphi = (\partial_\mu - igA_\mu)\varphi \qquad A_\mu = A_\mu^a T^a$$

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$$D_\mu \varphi = (\partial_\mu - igA_\mu)\varphi \qquad A_\mu = A_\mu^a T^a$$

$$\varphi \to \varphi' = U\varphi \qquad A_\mu \to A'_\mu \qquad D_\mu \varphi \to D'_\mu \varphi' = U D_\mu \varphi$$

$$(\partial_\mu - igA'_\mu)U\varphi = U(\partial_\mu - igA_\mu)\varphi$$

$$\partial_\mu U - igA'_\mu U = -igUA_\mu$$

$$A'_\mu = UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}$$

Infinitesimal tramsformation

$$\varphi(x) \to \varphi'(x) = (1 + i\alpha^a(x)T^a)\varphi(x)$$

$$A_\mu^a(x) \to A_\mu'^a(x) = A_\mu^a(x) + \frac{1}{g}D_\mu^{ab}\alpha^b(x)$$

$$D_\mu^{ab} = \delta^{ab}\partial_\mu - ig(T^c)^{ab}A_\mu^c$$

$$(T^c)^{ab} = if^{acb} \qquad [T^a, T^b] = if^{abc}T^c$$

Field strength

$$
\begin{aligned}
&[D_\mu, D_\nu]\varphi \\
&= \partial_\mu\partial_\nu\varphi - ig(\partial_\mu A_\nu)\varphi - igA_\nu\partial_\mu\varphi - igA_\mu\partial_\nu\varphi - g^2 A_\mu A_\nu\varphi \\
&- \partial_\nu\partial_\mu\varphi + ig(\partial_\nu A_\mu)\varphi + igA_\mu\partial_\nu\varphi + igA_\nu\partial_\mu\varphi + g^2 A_\nu A_\mu\varphi \\
&= -igG_{\mu\nu}\varphi \\
&G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] = G^a_{\mu\nu}T^a \\
&G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu
\end{aligned}
$$

$$G_{\mu\nu} \to UG_{\mu\nu}U^{-1} \qquad G^a_{\mu\nu} \to U^{ab}G^b_{\mu\nu}$$

Quantization

$$L = \left(D_\mu \varphi\right)^+ \left(D^\mu \varphi\right) - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}$$

Gauge

$$G^a(A) = \frac{1}{\xi} \partial^\mu A^a_\mu$$

$$L_\xi = -\frac{1}{2\xi} \left(\partial^\mu A^a_\mu\right)^2$$

Quantization

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$$G^a(A) = \frac{1}{\xi} \partial^\mu A^a_\mu$$

$$L_\xi = -\frac{1}{2\xi} \left(\partial^\mu A^a_\mu \right)^2$$

$$\delta G^a = \frac{1}{g} \partial^\mu D^{ab}_\mu \alpha^b$$

$$L_c = (\partial_\mu \bar{c}) (D^\mu c)$$

Standard Model

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

$$L = (\partial_\mu \varphi)^+ (\partial^\mu \varphi) - V(\varphi)$$

$SU(2) \times U(1)$

$$\varphi \to U\varphi \approx (1 + i\alpha^a T^a)\varphi$$

$$\varphi \to e^{i\beta}\varphi \approx (1 + i\beta)\varphi$$

Standard Model

$$\varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$
$$L = (\partial_\mu \varphi)^+ (\partial^\mu \varphi) - V(\varphi)$$

$SU(2) \times U(1)$

$$\varphi \to U\varphi \approx (1 + i\alpha^a T^a)\varphi$$
$$\varphi \to e^{i\beta}\varphi \approx (1 + i\beta)\varphi$$

Make both symmetries local ($Y = \frac{1}{2}$)

$$D_\mu \varphi = \left(\partial_\mu - ig_2 A_\mu^a T^a - ig_1 Y B_\mu\right)\varphi$$

$$L = \left(D_\mu \varphi\right)^+ \left(D^\mu \varphi\right) - V(\varphi) - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_2 \varepsilon^{abc} A^b_\mu A^c_\nu \qquad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

symmetric with respect to

$$\varphi \to (1 + i\alpha^a T^a + iY\beta)\varphi$$

$$A^a_\mu \to A^a_\mu + \frac{1}{g_2} D^{ab}_\mu \alpha^b \qquad D^{ab}_\mu = \delta^{ab} \partial_\mu + g_2 \varepsilon^{acb} A^c_\mu$$

$$B_\mu \to B_\mu + \frac{1}{g_1} \partial_\mu \beta$$

$$L = (D_\mu \varphi)^+ (D^\mu \varphi) - V(\varphi) - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_2 \varepsilon^{abc} A^b_\mu A^c_\nu \qquad B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

symmetric with respect to

$$\varphi \to (1 + i\alpha^a T^a + iY\beta)\varphi$$

$$A^a_\mu \to A^a_\mu + \frac{1}{g_2} D^{ab}_\mu \alpha^b \qquad D^{ab}_\mu = \delta^{ab} \partial_\mu + g_2 \varepsilon^{acb} A^c_\mu$$

$$B_\mu \to B_\mu + \frac{1}{g_1} \partial_\mu \beta$$

Vacuum expectation value

$$<\varphi> = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}$$

symmetric with respect to $U(1)$ $\alpha^1 = \alpha^2 = 0, \alpha^3 = \beta$

Unitary gauge

Make a gauge transformation

$$\varphi = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v + h \end{pmatrix}$$

Gauge bosons

$$W^{\pm}_{\mu} = \frac{1}{\sqrt{2}}\left(A^1_\mu \mp iA^2_\mu\right)$$
$$W^{\pm}_{\mu\nu} = \partial_\mu W^{\pm}_\nu - \partial_\nu W^{\pm}_\mu$$
$$A^3_{\mu\nu} = \partial_\mu A^3_\nu - \partial_\nu A^3_\mu$$

Kinetic terms

$$L_k = -\frac{1}{2}W^{+}_{\mu\nu}W^{-\mu\nu} - \frac{1}{4}A^3_{\mu\nu}A^{3\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

Gauge bosons

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(A_\mu^1 \mp iA_\mu^2\right)$$
$$W_{\mu\nu}^\pm = \partial_\mu W_\nu^\pm - \partial_\nu W_\mu^\pm$$
$$A_{\mu\nu}^3 = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3$$

Kinetic terms

$$L_k = -\frac{1}{2}W_{\mu\nu}^+ W^{-\mu\nu} - \frac{1}{4}A_{\mu\nu}^3 A^{3\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu}$$

$$D_\mu\varphi = -i\left\{\frac{g_2 v}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\left(W_\mu^+T^+ + W_\mu^-T^-\right) + A_\mu^3 T^3\right] + \frac{g_1 v}{2\sqrt{2}}B_\mu\right\}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \cdots$$

$$T^\pm = T^1 \pm iT^2 \qquad T^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad T^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$D_\mu \varphi = i\frac{v}{2} \begin{pmatrix} -g_2 W_\mu^+ \\ \frac{1}{\sqrt{2}} \left(g_2 A_\mu^3 - g_1 B_\mu \right) \end{pmatrix} + \cdots$$

$$D_\mu \varphi = i\frac{v}{2} \begin{pmatrix} -g_2 W_\mu^+ \\ \frac{1}{\sqrt{2}} \left(g_2 A_\mu^3 - g_1 B_\mu \right) \end{pmatrix} + \cdots$$

Mass terms

$$L_m = \frac{g_2^2 v^2}{4} W_\mu^+ W^{-\mu} + \frac{\left(g_1^2 + g_2^2\right) v^2}{8} \left(A_\mu^3 \cos\theta_w - B_\mu \sin\theta_w \right)^2$$

where

$$\cos\theta_w = \frac{g_2}{\sqrt{g_1^2 + g_2^2}} \qquad \sin\theta_w = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}$$

$$Z_\mu = A_\mu^3 \cos\theta_w - B_\mu \sin\theta_w$$
$$A_\mu = A_\mu^3 \sin\theta_w + B_\mu \cos\theta_w$$

$$\begin{aligned} L_0 = &-\frac{1}{2} W_{\mu\nu}^+ W^{-\mu\nu} + m_W^2 W_\mu^+ W^{-\mu} \\ &- \frac{1}{4} Z_{\mu\nu} Z^{\mu\nu} + \frac{m_Z^2}{2} Z_\mu Z^\mu \\ &- \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \end{aligned}$$

where

$$Z_{\mu\nu} = \partial_\mu Z_\nu - \partial_\nu Z_\mu \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

and

$$m_W = \frac{g_2 v}{2} \qquad m_Z = \frac{\sqrt{g_1^2 + g_2^2}\, v}{2} \qquad m_W = m_Z \cos\theta_w$$

Higgs boson

$$L_h = \frac{1}{2}\left(\partial_\mu h\right)\left(\partial^\mu h\right) - V(\varphi)$$

$$V(\varphi) = \frac{\lambda}{4}\left(\varphi^+\varphi - \frac{v^2}{2}\right)^2 = \frac{\lambda v^2}{4}h^2 + \frac{\lambda v}{4}h^3 + \frac{\lambda}{16}h^4$$

$$m_h^2 = \frac{\lambda v^2}{2}$$

Higgs – gauge bosons interactions

$$\frac{g_2^2(v+h)^2}{4} W_\mu^+ W^{-\mu} = m_W^2 \left(1 + \frac{h}{v}\right)^2 W_\mu^+ W^{-\mu}$$

interactions

$$2\frac{m_W^2}{v} h W_\mu^+ W^{-\mu} + \frac{m_W^2}{v^2} h^2 W_\mu^+ W^{-\mu} = m_W g_2 h W_\mu^+ W^{-\mu} + \frac{g_2^2}{4} h^2 W_\mu^+ W^-$$

Higgs – gauge bosons interactions

$$\frac{g_2^2(v+h)^2}{4}W_\mu^+W^{-\mu} = m_W^2\left(1+\frac{h}{v}\right)^2 W_\mu^+W^{-\mu}$$

interactions

$$2\frac{m_W^2}{v}hW_\mu^+W^{-\mu}+\frac{m_W^2}{v^2}h^2W_\mu^+W^{-\mu} = m_W g_2 hW_\mu^+W^{-\mu}+\frac{g_2^2}{4}h^2W_\mu^+W^{-}$$

$$\frac{\left(g_1^2+g_2^2\right)(v+h)^2}{8}Z_\mu Z^\mu = m_Z^2\left(1+\frac{h}{v}\right)^2 Z_\mu Z^\mu$$

interactions

$$2\frac{m_Z^2}{v}hZ_\mu Z^\mu+\frac{m_Z^2}{v^2}h^2Z_\mu Z^\mu = m_Z\sqrt{g_1^2+g_2^2}hZ_\mu Z^\mu+\frac{g_1^2+g_2^2}{4}h^2Z_\mu Z^\mu$$

Gauge-bosons interactions

$$
\begin{aligned}
-\frac{1}{4}G^a_{\mu\nu}G^{a\mu\nu} = {} & \text{(quadratic)} \\
& - g_2\varepsilon^{abc}\left(\partial_\mu A^a_\nu\right)A^b_\mu A^c_\nu - \frac{g_2^2}{4}\varepsilon^{abc}\varepsilon^{ade}A^b_\mu A^c_\nu A^d_\mu A^e_\nu \\
= {} & \text{(quadratic)} \\
& - g_2\varepsilon^{abc}\left(\partial_\mu A^a_\nu\right)A^b_\mu A^c_\nu - \frac{g_2^2}{4}\left[\left(A^a_\mu A^{a\mu}\right)^2 - A^a_\mu A^a_\nu A^b_\mu A^b_\nu\right]
\end{aligned}
$$

3-boson vertices

$$
\begin{aligned}
L_3 = i g_2 \bigg[\frac{1}{2} \left(\partial_\mu A^3_\nu\right) \left(W^+_\mu W^-_\nu - W^-_\mu W^+_\nu\right) \\
+ \left(\partial_\mu W^+_\nu\right) \left(W^-_\mu A^3_\nu - A^3_\mu W^-_\nu\right) \\
+ \left(\partial_\mu W^-_\nu\right) \left(A^3_\mu W^+_\nu - W^+_\mu A^3_\nu\right) \bigg]
\end{aligned}
$$

where

$$
A^3_\mu = Z_\mu \cos\theta_w + A_\mu \sin\theta_w
$$

4-boson vertices

$$
\begin{aligned}
L_4 = \frac{g_2^2}{2} \Big[W^+_\mu W^{+\mu} W^-_\nu W^{-\nu} - \left(W^+_\mu W^{-\mu}\right)^2 \\
+ 2 W^+_\mu A^{3\mu} W^-_\nu A^{3\nu} - 2 W^+_\mu W^{-\mu} A^3_\nu A^{3\nu} \Big]
\end{aligned}
$$

Fermions

$$
\begin{aligned}
L &= \bar{\psi} i \gamma^\mu D_\mu \psi \\
D_\mu &= \partial_\mu - i g_2 A_\mu^a T^a - i g_1 Y B_\mu \\
&= \partial_\mu - i \frac{g_2}{\sqrt{2}} \left(W_\mu^+ T^+ + W_\mu^- T^- \right) \\
&- ieQA_\mu - i \sqrt{g_1^2 + g_2^2} \left(T^3 - Q \sin^2 \theta_w \right) Z_\mu
\end{aligned}
$$

where

$$
e = \frac{g_1 g_2}{\sqrt{g_1^2 + g_2^2}} \qquad Q = T^3 + Y
$$

1 generation of leptons

$$l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \quad \nu_R \quad e_R$$

$$Y = -\frac{1}{2} \qquad 0 \quad -1$$

ν_R does not interact

1 generation of leptons

$$l_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \quad \nu_R \quad e_R$$

$$Y = -\frac{1}{2} \qquad 0 \quad -1$$

ν_R does not interact

Cannot write mass terms, but

$$L_y = -y\bar{l}_L \varphi e_R + \text{h.c.} = -\frac{y}{\sqrt{2}}(v+h)\left(\bar{e}_L e_R + \bar{e}_R e_L\right)$$

gives

$$m_e = \frac{yv}{\sqrt{2}}$$

and interaction with Higgs

$$-\frac{m_e}{v} h \bar{e} e$$

Neutrino mass

$$\tilde{\varphi}_i = \varepsilon_{ij}\varphi^i \qquad \tilde{\varphi} = \varphi^T \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$L_{y\nu} = -y_\nu \bar{\nu}_R \tilde{\varphi} l_L + \text{h.c.} = -\frac{y_\nu}{\sqrt{2}}(v+h)\left(\bar{\nu}_R \nu_L + \bar{\nu}_L \nu_R\right)$$

$$m_\nu = \frac{y_\nu v}{\sqrt{2}}$$

1 generation of quarks

$$
\begin{array}{rccc}
q_L = & \begin{pmatrix} u_L \\ d_L \end{pmatrix} & u_R & d_R \\
Y = & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{3}
\end{array}
$$

Color $SU(3)$, gluons

1 generation of quarks

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} \quad u_R \quad d_R$$
$$Y = -\frac{1}{6} \qquad \frac{2}{3} \quad -\frac{1}{3}$$

Color $SU(3)$, gluons

$$L_y = -y_d \bar{q}_L \varphi d_R - y_u \bar{u}_R \tilde{\varphi} q_L + \text{h.c.}$$
$$m_u = \frac{y_u v}{\sqrt{2}} \qquad m_d = \frac{y_d v}{\sqrt{2}}$$

Interaction with Higgs

$$-\frac{m_u}{v} h \bar{u} u - \frac{m_d}{v} h \bar{d} d$$