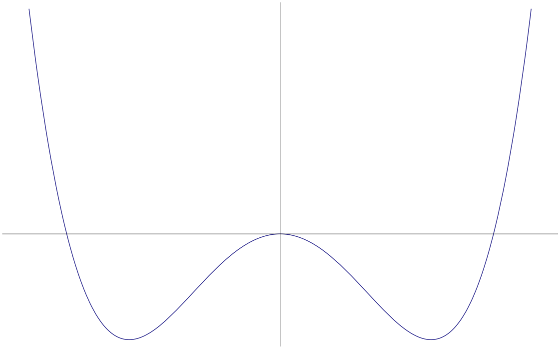
Spontaneous symmetry breaking

$$m^2 \to -\mu^2$$

$$L = \frac{1}{2}\left(\partial_\mu \varphi\right)\left(\partial^\mu \varphi\right) - V(\varphi)$$

$$V(\varphi) = -\frac{\mu^2}{2}\varphi^2 + \frac{\lambda}{4!}\varphi^4 = \frac{\lambda}{4!}(\varphi^2 - v^2)^2 + \text{const} \qquad v^2 = 3!\frac{\mu^2}{\lambda}$$

Out[5]=



Quantum mechanics

Ground state is even, splitting $\sim$ tunneling amplitude

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Quantum field theory

$N \to \infty$ large space regions $\gg$ correlation length

Tunneling amplitude

$$A^N \to 0$$

2 degenerate vacua, each breaks the $\varphi \to -\varphi$ symmetry

Quantum mechanics

Ground state is even, splitting $\sim$ tunneling amplitude

Quantum field theory

$N \to \infty$ large space regions $\gg$ correlation length

Tunneling amplitude

$$A^N \to 0$$

2 degenerate vacua, each breaks the $\varphi \to -\varphi$ symmetry

Suppose we are in the $<\varphi> = +v$ vacuum

$$\varphi = v + h$$

$$L = \frac{1}{2}\left(\partial_\mu h\right)\left(\partial^\mu h\right) - \mu^2 h^2 - \frac{\lambda v}{3!} h^3 - \frac{\lambda}{4!} h^4$$

$m = \sqrt{2}\mu$, triple and quadruple vertices

Renormalization

$$\varphi_0 = v_0 + h_0 \qquad (<h_0> = 0)$$

$$L = \frac{1}{2}\left(\partial_\mu h_0\right)\left(\partial^\mu h_0\right) - \frac{m_0^2}{2} h_0^2$$
$$+ \left(\mu_0^2 - \frac{\lambda_0}{3!} v_0^2\right) v_0 h_0 - \frac{\lambda_0 v_0}{3!} h_0^3 - \frac{\lambda_0}{4!} h_0^4$$

$$m_0^2 = \frac{\lambda_0}{2} v_0^2 - \mu_0^2$$

Renormalization

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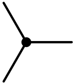
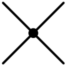
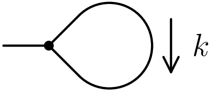
$$m_0^2 = \frac{\lambda_0}{2}v_0^2 - \mu_0^2$$


$$= i\left(\mu_0^2 - \frac{\lambda_0}{3!}v_0^2\right) v_0$$


$$= -i\lambda_0 v_0$$


$$= -i\lambda_0$$

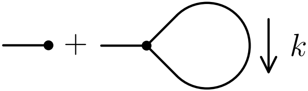
Vacuum expectation value

$<h_0> = 0 =$  $+$ 

$$= i\left(\mu_0^2 - \frac{\lambda_0}{3!}v_0^2\right)v_0 - \frac{\lambda_0 v_0}{2}\int \frac{d^d k}{(2\pi)^d}\frac{1}{m_0^2 - k^2 - i0}$$

$$\frac{\lambda_0}{3!}v_0^2 - \mu_0^2 = \frac{m_0^2}{d-2}\frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)$$

Vacuum expectation value

$<h_0> = 0 =$ 

$$= i\left(\mu_0^2 - \frac{\lambda_0}{3!}v_0^2\right)v_0 - \frac{\lambda_0 v_0}{2}\int \frac{d^d k}{(2\pi)^d}\frac{1}{m_0^2 - k^2 - i0}$$

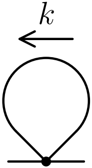
$$\frac{\lambda_0}{3!}v_0^2 - \mu_0^2 = \frac{m_0^2}{d-2}\frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)$$

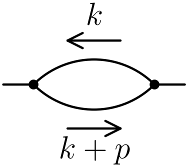
In the correction

$$v_0^2 = 3!\frac{\mu_0^2}{\lambda_0} \qquad m_0^2 = 2\mu_0^2$$

$$v_0^2 = 3!\frac{\mu_0^2}{\lambda_0}\left[1 + \frac{2}{d-2}\frac{\lambda_0(2\mu_0^2)^{-\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon) + \cdots\right]$$

Self energy





$$\Sigma(p) = -\frac{\lambda_0 m_0^{d-2}}{(4\pi)^{d/2}} \left[\frac{\Gamma(\varepsilon)}{d-2} + \frac{3}{2} f\left(\frac{p^2}{m_0^2} \right) \right]$$

On-shell mass and field renormalization

Mass

$$m_0^2 = m_{\text{os}}^2 \left[1 + \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \left(\frac{\Gamma(\varepsilon)}{d-2} + \frac{3}{2} f(1) \right) + \cdots \right]$$

$$= m_{\text{os}}^2 \left[1 + \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \left(2 + \frac{\varepsilon}{2} + \frac{3}{2} \left(f(1) - f(0) \right) \varepsilon + \mathcal{O}(\varepsilon^2) \right) + \cdot$$

$\left(f(1) - f(0) \right)_{\varepsilon=0}$ is a number

On-shell mass and field renormalization

Mass

$$m_0^2 = m_{\text{os}}^2 \left[1 + \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \left(\frac{\Gamma(\varepsilon)}{d-2} + \frac{3}{2} f(1)\right) + \cdots\right]$$

$$= m_{\text{os}}^2 \left[1 + \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \left(2 + \frac{\varepsilon}{2} + \frac{3}{2}\left(f(1) - f(0)\right)\varepsilon + \mathcal{O}(\varepsilon^2)\right) + \cdot$$

$\left(f(1) - f(0)\right)_{\varepsilon=0}$ is a number

Field $p^2 \to m_{\text{os}}^2$

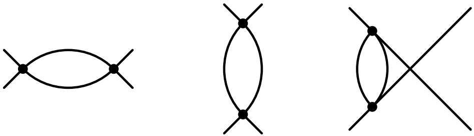
$$G(p) = \frac{1}{p^2 - m_0^2 - \Sigma(p^2) + i0} = \frac{Z_h^{\text{os}}}{p^2 - m_{\text{os}}^2 + i0}$$

$$Z_h^{\text{os}} = \frac{1}{1 - \left(\frac{d\Sigma(p^2)}{dp^2}\right)_{p^2 = m_{\text{os}}^2}} = 1 - \frac{3}{2} \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} f'(1) + \cdots$$

$f'(1)$ is finite at $\varepsilon \to 0$

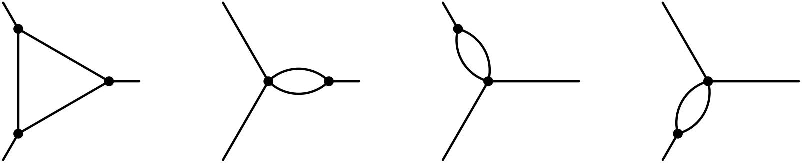
4-leg vertex

UV divergences only in



$\overline{\text{MS}}$ renormalization of λ –
the same as in the unbroken theory

3-leg vertex



First – convergent

$$\Gamma = 1 - \frac{1}{2}\frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}}\left[f\left(\frac{p_1^2}{m_0^2}\right) + f\left(\frac{p_2^2}{m_0^2}\right) + f\left(\frac{p_3^2}{m_0^2}\right) + \text{finite}\right]$$
$$= 1 - \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\left(\frac{3}{2} + \mathcal{O}(\varepsilon)\right)$$

Gives the same renormalization of λ

Complex scalar field

$$L = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi - \frac{\lambda}{4} (\varphi^* \varphi)^2$$

$U(1)$ symmetry $\varphi \to e^{i\alpha} \varphi$

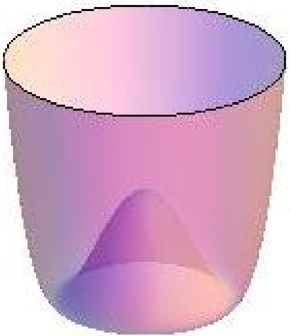
Spontaneous symmetry breaking

$$m^2 \to -\mu^2$$

$$L = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - V(\varphi)$$

$$V(\varphi) = -\mu^2 \varphi^* \varphi + \frac{\lambda}{4}(\varphi^* \varphi)^2 = \frac{\lambda}{4}\left(\varphi^* \varphi - \frac{v^2}{2}\right)^2 + \text{const}$$

$$v^2 = 4\frac{\mu^2}{\lambda}$$



Vacua

$$<\theta|\varphi|\theta> = \frac{v}{\sqrt{2}}e^{i\theta}$$

Suppose we are in the $\theta = 0$ vacuum

$$\varphi = \frac{1}{\sqrt{2}}(v + h + ib)$$

$$L = \frac{1}{2}\left(\partial_\mu h\right)\left(\partial^\mu h\right) + \frac{1}{2}\left(\partial_\mu b\right)\left(\partial^\mu b\right) - \mu^2 h^2 - \frac{\lambda v}{4}h(h^2 + b^2) - \frac{\lambda}{16}(h^2 + b^2)^2$$

$$v = \frac{2\mu}{\sqrt{\lambda}}$$

h – $m = \sqrt{2}\mu$

b – massless Goldstone boson

Polar coordinates

$$\varphi = \frac{1}{\sqrt{2}}(v+h)e^{i\theta/v}$$

$$L = \frac{1}{2}\left(\partial_\mu h\right)\left(\partial^\mu h\right) + \frac{1}{2}\left(1+\frac{h}{v}\right)^2\left(\partial_\mu \theta\right)\left(\partial^\mu \theta\right) - \mu^2 h^2 - \frac{\lambda v}{4}h^3 - \frac{\lambda}{16}h^4$$

h – $m = \sqrt{2}\mu$

θ – massless Goldstone boson

Polar coordinates

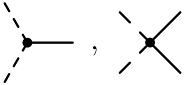
$$\varphi = \frac{1}{\sqrt{2}}(v + h)e^{i\theta/v}$$

$$L = \frac{1}{2}\left(\partial_\mu h\right)\left(\partial^\mu h\right) + \frac{1}{2}\left(1 + \frac{h}{v}\right)^2 \left(\partial_\mu \theta\right)\left(\partial^\mu \theta\right) - \mu^2 h^2$$

$$- \frac{\lambda v}{4} h^3 - \frac{\lambda}{16} h^4$$

h – $m = \sqrt{2}\mu$

θ – massless Goldstone boson



$\sim p_1 \cdot p_2$

$$\Sigma_\theta(p \to 0) \to 0$$

θ remains massless

Jacobian

$$\prod_x (v + h(x)) \sim \int e^{iS_c} D\bar{c}\, Dc$$

$$S_c = \int L_c d^4x \qquad L_c = -m_c^2 \bar{c} \left(1 + \frac{h}{v}\right) c$$

c – scalar fermion field (ghost), propagator

$$G_c(p) = -\frac{1}{m_c^2}$$

Jacobian

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c – scalar fermion field (ghost), propagator

$$G_c(p) = -\frac{1}{m_c^2}$$

Anticommuting variables (always dimensionless)

$$c_i c_j + c_j c_i = 0 \qquad c_i^2 = 0$$

$$\int dc = 0 \qquad \int c\, dc = 1$$

$$e^{-a\bar{c}c} = 1 - a\bar{c}c \qquad \int e^{-a\bar{c}c}\, d\bar{c}\, dc = a$$

Goldstone theorem

N real fields

$$L = \frac{1}{2}\left(\partial_\mu \varphi_i\right)\left(\partial^\mu \varphi_i\right) - V(\varphi) \qquad V(\varphi) = \frac{m^2}{2}\varphi_i\varphi_i + \frac{\lambda}{4!}\left(\varphi_i\varphi_i\right)^2$$

$SO(N)$ symmetry, $N(N-1)/2$ generators
(rotations in coordinate planes)

Goldstone theorem

N real fields

$$L = \frac{1}{2} \left(\partial_\mu \varphi_i\right) \left(\partial^\mu \varphi_i\right) - V(\varphi) \qquad V(\varphi) = \frac{m^2}{2} \varphi_i \varphi_i + \frac{\lambda}{4!} \left(\varphi_i \varphi_i\right)^2$$

$SO(N)$ symmetry, $N(N-1)/2$ generators
(rotations in coordinate planes)
$m^2 \to -\mu^2$

$$V(\varphi) = \frac{\lambda}{4!} \left(\varphi_i \varphi_i - v^2\right)^2$$

Suppose we are in the vacuum $<\varphi_i> = v\delta_{iN}$
$SO(N-1)$ symmetry (with $(N-1)(N-2)/2$ generators) remains;
$N-1$ generators are broken
$N-1$ flat directions $\Rightarrow$ massless Goldstone bosons

Scalar electrodynamics

$$L = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi - \frac{\lambda}{4} (\varphi^* \varphi)^2$$

Symmetry $\varphi \to e^{i\alpha}\varphi$, $\alpha = \text{const.}$ $\alpha(x)$?

Scalar electrodynamics

$$L = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi - \frac{\lambda}{4} (\varphi^* \varphi)^2$$

Symmetry $\varphi \to e^{i\alpha}\varphi$, $\alpha = \text{const.}$ $\alpha(x)$?

$$D_\mu \varphi = (\partial_\mu - ieA_\mu)\varphi \qquad D_\mu \varphi \to e^{i\alpha} D_\mu \varphi$$

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha$$

Scalar electrodynamics

$$L = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi - \frac{\lambda}{4} (\varphi^* \varphi)^2$$

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$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha$$

$$[D_\mu, D_\nu]\varphi \to e^{i\alpha}[D_\mu, D_\nu]\varphi$$

$$[D_\mu, D_\nu]\varphi = -ieF_{\mu\nu}\varphi \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Scalar electrodynamics

$$L = (\partial_\mu \varphi^*)(\partial^\mu \varphi) - m^2 \varphi^* \varphi - \frac{\lambda}{4} (\varphi^* \varphi)^2$$

Symmetry $\varphi \to e^{i\alpha}\varphi$, $\alpha = \text{const.}$ $\alpha(x)$?

$$D_\mu \varphi = (\partial_\mu - ieA_\mu)\varphi \qquad D_\mu \varphi \to e^{i\alpha} D_\mu \varphi$$

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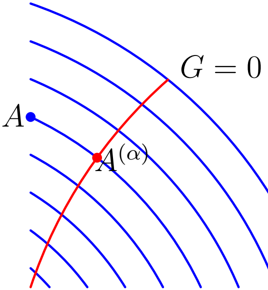
$$[D_\mu, D_\nu]\varphi = -ieF_{\mu\nu}\varphi \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$L = (D_\mu \varphi)^* (D^\mu \varphi) - m^2 \varphi^* \varphi - \frac{\lambda}{4} (\varphi^* \varphi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

invariand with respect to

$$\varphi(x) \to e^{i\alpha(x)}\varphi(x) \qquad A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$$

Gauge fields



Orbits of the gauge group $A \to A^{(\alpha)}$
Gauge $G(A^{(\alpha)}(x)) = 0$ — unique solution $\alpha(x)$
for any given $A(x)$
The surface $G = 0$ intersects any orbit at 1 point

Faddeev–Popov determinant

$$\Delta^{-1}(A) = \int \prod_x \delta(G(A^{(\alpha)}(x)))\, D\alpha$$

Near the surface $G(A^{(\alpha_0)}) = 0$:

$$\delta G(A(x)) = \hat{M}\delta\alpha(x)$$

$$\Delta^{-1}(A) = \int \delta(\hat{M}\alpha(x))\, D\alpha = 1/\det \hat{M}$$

Faddeev–Popov determinant

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$$\Delta^{-1}(A) = \int \delta(\hat{M}\alpha(x))\, D\alpha = 1/\det \hat{M}$$

$$Z(J) = \int e^{iS(A)}\, DA$$

$$= \int \Delta(A) \prod_x \delta(G(A^{(\alpha)}(x)))\, e^{iS(A)}\, D\alpha\, DA$$

$$= \left(\prod_x \int d\alpha\right) \times \int \Delta(A) \prod_x \delta(G(A(x)))\, e^{iS(A)}\, DA$$

Ghosts

$$\Delta(A) = \det \hat{M} = \int e^{iS_c} \, D\bar{c} \, Dc$$

$$S_c = \int L_c d^4x \qquad L_c \sim \bar{c}\hat{M}c$$

Ghosts

$$\Delta(A) = \det \hat{M} = \int e^{iS_c}\, D\bar{c}\, Dc$$

$$S_c = \int L_c d^4x \qquad L_c \sim \bar{c}\hat{M}c$$

Generalized Lorenz gauge $G(A(x)) = \partial_\mu A^\mu(x) - \omega(x)$

$$\delta G(x) = \partial_\mu \delta A^\mu(x) = \frac{1}{e}\partial_\mu \partial^\mu \delta\alpha(x)$$

$$\hat{M} = \frac{1}{e}\partial_\mu \partial^\mu \qquad \det \hat{M} = \text{const}$$

$$L_c = -\bar{c}\partial_\mu \partial^\mu c \Rightarrow (\partial_\mu \bar{c})\,(\partial^\mu c)$$

Covariant gauge

$$Z(J) = \int \prod_x \delta(\partial_\mu A^\mu(x) - \omega(x))\, e^{iS}\, DA$$

Covariant gauge

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Covariant gauge

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$$\times\, e^{-\frac{i}{2\xi}\int \omega^2(x)\, d^4x}\, D\omega$$

$$= \int e^{i(S+S_\xi)}\, DA \qquad L_\xi = -\frac{1}{2\xi}\,(\partial_\mu A^\mu)^2$$

Photon propagator

$$L = -\frac{1}{2}\left(\partial_\mu A_\nu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)\left(\partial_\nu A^\nu\right) \to \frac{1}{2}A^\mu M_{\mu\nu} A^\nu$$

$$M_{\mu\nu}(\partial) = g_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu + \frac{1}{\xi}\partial_\mu\partial_\nu$$

$$M_{\mu\nu}(-ip) = -p^2\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} + \frac{1}{\xi}\frac{p_\mu p_\nu}{p^2}\right]$$

$$M^{-1}_{\mu\nu}(-ip) = -\frac{1}{p^2}\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} + \xi\frac{p_\mu p_\nu}{p^2}\right]$$

Photon propagator

$$L = -\frac{1}{2}\left(\partial_\mu A_\nu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - \frac{1}{2\xi}\left(\partial_\mu A^\mu\right)\left(\partial_\nu A^\nu\right) \to \frac{1}{2}A^\mu M_{\mu\nu} A^\nu$$

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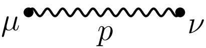

$$= -iD^0_{\mu\nu}(p)$$

$$D^0_{\mu\nu}(p) = \frac{1}{p^2}\left[g_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right]$$