

Energy levels of a hydrogen-like atom in a superstrong magnetic field

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(Phys. Rev. D 85, 044058 (2012), <http://arxiv.org/abs/1112.1891>)

Baikal School-2012

July 10, 2012

- Potential of the pointlike charge in a superstrong magnetic field
 - One loop contribution
 - Potential along the magnetic field
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- Energy levels in the modified potential
 - Energy levels of the hydrogen atom with the account of screening (Schrödinger)
 - Dirac equation
 - Used methods
 - Results for $Z = 1$
 - Critical nucleus charge
 - Results for $Z = 40$
 - Critical nucleus charge with the account of screening
- Conclusions

$$\hbar = c = 1$$

Gauss units:

$$e^2 = \alpha = 1/137.03599\dots$$

Magnetic fields:

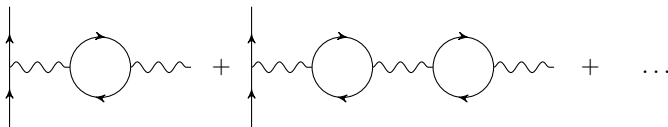
$$B_a = m_e^2 e^3 = 2.3 \cdot 10^9 G$$

$$B_0 = m_e^2 / e = 4.4 \cdot 10^{13} G$$

One loop contribution

(A.E.Shabad, V.V.Usov, *Phys. Rev. D.* 77 (2008), 025001)

(B.Machet, M.I.Vysotsky, *Phys. Rev. D* 83 (2011), 025022)

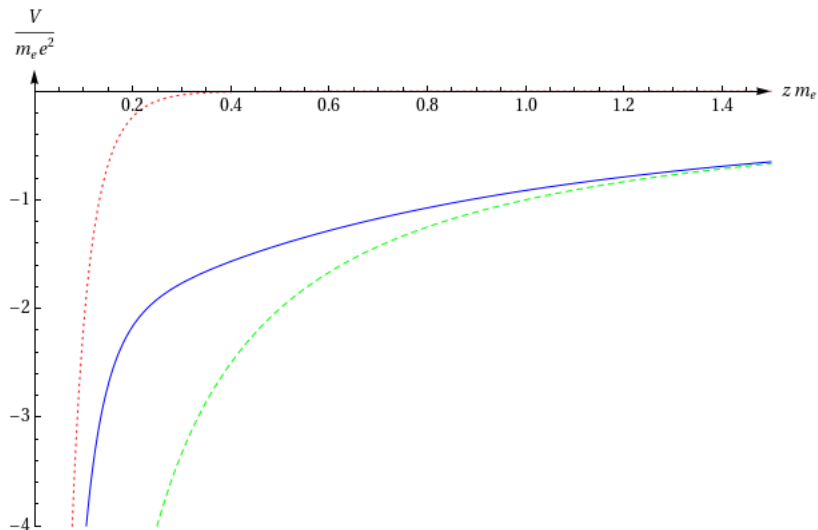


$$\Phi(z, \rho) = 4\pi e \int \frac{e^{i\vec{k}_\perp \bar{\rho} + ik_\parallel z} dk_\parallel d^2 k_\perp / (2\pi)^3}{k_\parallel^2 + k_\perp^2 + \frac{2e^3 B}{\pi} \exp(-\frac{k_\perp^2}{2eB}) \frac{k_\parallel^2}{6m_e^2 + k_\parallel^2}}$$

$$\Phi(z, 0) = \frac{e}{|z|} \left[1 - e^{-\sqrt{6m_e^2}|z|} + e^{-\sqrt{(2/\pi)e^3 B + 6m_e^2}|z|} \right] .$$

$$\Phi(z, 0) = \begin{cases} \frac{e}{|z|} e^{-\sqrt{(2/\pi)e^3 B}|z|} & , \quad |z| < \frac{1}{\sqrt{(2/\pi)e^3 B}} \ln \left(\sqrt{\frac{e^3 B}{3\pi m_e^2}} \right) \\ \frac{e}{|z|} \left(1 - e^{-\sqrt{6m_e^2}|z|} \right) & , \quad \frac{1}{\sqrt{(2/\pi)e^3 B}} \ln \left(\sqrt{\frac{e^3 B}{3\pi m_e^2}} \right) < |z| < \frac{1}{m_e} \\ \frac{e}{|z|} & , \quad \frac{1}{m_e} < |z| \end{cases}$$

Modified potential along z axis ($B = 5 \cdot 10^4 B_0$)



(Numerically obtained in the paper A.E. Shabad, V.V. Usov, *Phys. Rev.D*, 77, 025001)

$$\Phi(z, \rho) = 4\pi e \int \frac{e^{i\vec{k}_\perp \cdot \vec{\rho} + ik_\parallel z} dk_\parallel d^2 k_\perp / (2\pi)^3}{k_\parallel^2 + k_\perp^2 + \frac{2e^3 B}{\pi} \exp(-\frac{k_\perp^2}{2eB}) \frac{k_\parallel^2}{6m_e^2 + k_\parallel^2}} .$$

↓ at $B \gg m^2/e^3$

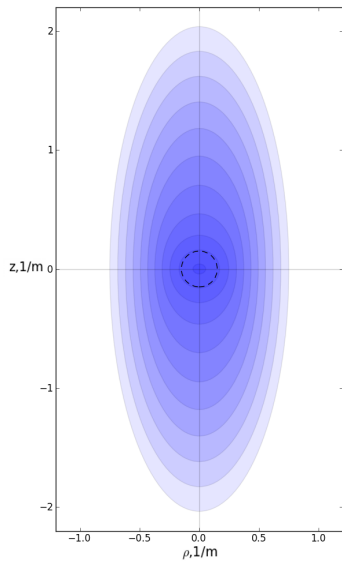
$$\Phi(0, \rho) = \begin{cases} \frac{e}{\rho} \exp(-\sqrt{(2/\pi)e^3 B} \rho) & , \quad \rho < \frac{1}{\sqrt{(2/\pi)e^3 B}} \ln \sqrt{\frac{e^3 B}{3\pi m_e^2}} \\ \sqrt{\frac{3\pi m_e^2}{e^3 B}} \frac{e}{\rho} & , \quad \frac{1}{\sqrt{(2/\pi)e^3 B}} \ln \sqrt{\frac{e^3 B}{3\pi m_e^2}} < \rho \end{cases} ,$$

At large distances $z \gg 1/m_e$ ($k_{\parallel} \ll m_e$)

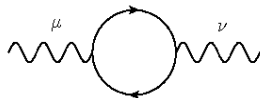
$$\begin{aligned} \Phi(\vec{\rho}, z) &= \int_{-\infty}^{+\infty} \frac{dk_{\parallel}}{2\pi} e^{ik_{\parallel}z} \int \frac{d^2k_{\perp}}{(2\pi)^2} e^{i\vec{k}_{\perp}\vec{\rho}} \frac{4\pi e}{\left(1 + \frac{e^3 B}{3\pi m_e^2}\right) k_{\parallel}^2 + k_{\perp}^2} = \\ &= \frac{e}{\sqrt{z^2 + \left(1 + \frac{e^3 B}{3\pi m_e^2}\right) \rho^2}}. \end{aligned}$$

So equipotential lines are ellipses.

Equipotential lines



$$G(k) = e^{-k_{\perp}^2/eB} (1 - i\gamma_1\gamma_2) \frac{\hat{k}_{0,3} + m_e}{k_{0,3}^2 - m_e^2}$$

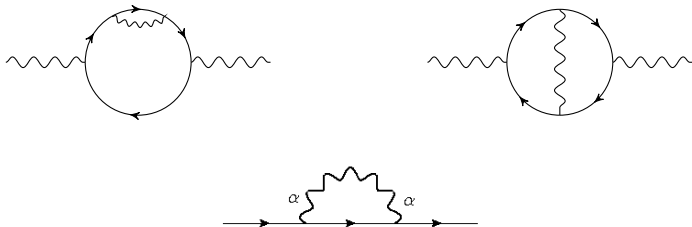


Integral over transverse momentum:

$$\int d^2k_{\perp} e^{-k_{\perp}^2/eB} \sim eB$$

As a result we have:

$$\Pi_{\mu\nu} \simeq e^3 B \exp\left(-\frac{q_{\perp}^2}{eB}\right) \Pi_{\mu\nu}^{(2)}(q_{\parallel})$$



Corresponding integral:

$$\int d^2 k_{\perp} d^2 k_{\parallel} e^{-k_{\perp}^2/eB} \gamma_{\alpha} (1 - i\gamma_1 \gamma_2) \frac{\hat{k}_{0,3} + m_e}{k_{0,3}^2 - m_e^2} \gamma_{\alpha} \frac{1}{(k - q)_{\parallel}^2 - (k - q)_{\perp}^2}$$

Matrix expression:

$$\gamma_{\mu} (1 - i\gamma_1 \gamma_2) \hat{k}_{0,3} \gamma_{\mu} = -2[\hat{k}_{0,3} - i\hat{k}_{0,3} \gamma_2 \gamma_1] = -2\hat{k}_{0,3} (1 + i\gamma_1 \gamma_2) ,$$

$$\text{But } (1 + i\gamma_1 \gamma_2)(1 - i\gamma_1 \gamma_2) = 0!$$

So the initial integral is proportional to m_e and not \sqrt{eB} .

Energy levels of hydrogen atom with the account of screening (Schrödinger)

(B.M. Karnakov, V.S. Popov, JETP, 97, 890)

Karnakov–Popov equation:

$$\ln\left(\frac{B}{B_a}\right) = \lambda + 2 \ln \lambda + 2\psi\left(1 - \frac{1}{\lambda}\right) + \ln 2 + 4\gamma + \psi(1 + |m|)$$

where λ defines corresponding energy $E_m = -(m_e e^4 / 2) \lambda^2$

(B.Machet, M.I.Vysotsky, Phys. Rev. D 83 (2011), 025022)

With the account of screening for ground energy level we get:

$$\ln\left(\frac{B/B_a}{1 + \frac{e^6 B}{3\pi B_a}}\right) = \lambda + 2 \ln \lambda + 2\psi\left(1 - \frac{1}{\lambda}\right) + \ln 2 + 3\gamma$$

For $B \rightarrow \infty$ we get $\lambda_\infty \approx 11.21$ or $E_0^\infty = -1.7 \text{keV}$.

For the fields $B \gg m^2 e^3 Z^2$ ($a_H = \frac{1}{\sqrt{eB}} \ll \frac{1}{m_e Z e^2} = a_B$)

$$(\alpha(\mathbf{p} + \mathbf{eA}) + V + \beta)\psi = \varepsilon\psi$$

$$\Downarrow$$

$$\frac{d^2\chi}{dz^2} + 2m_e(E - U)\chi = 0 ,$$

$$E = \frac{\varepsilon^2 - m_e^2}{2m_e},$$

$$U = \frac{\varepsilon}{m_e} \bar{V} - \frac{1}{2m_e} \bar{V}^2 + \frac{\bar{V}''}{4m_e(\varepsilon + m_e - V)} + \frac{3/8(\bar{V}')^2}{m_e(\varepsilon + m_e - V)^2} ,$$

$$\bar{V} = \frac{1}{a_H^2} \int_0^\infty V(\sqrt{\rho^2 + z^2}) \exp\left(-\frac{\rho^2}{2a_H^2}\right) \rho d\rho$$

Results for $Z = 1$ without taking screening into account

$$\bar{V}(z) = -\frac{Ze^2}{a_H} \sqrt{\frac{\pi}{2}} \exp\left(\frac{z^2}{2a_H^2}\right) \operatorname{erfc}\left(\frac{|z|}{\sqrt{2}a_H}\right)$$

$$E = \frac{\varepsilon^2 - m_e^2}{2m_e} \equiv -\frac{m_e e^4}{2} \lambda^2$$

For λ we get:

$\frac{B}{B_0}$	Karnakov–Popov equation (Schrödinger)	Numerical results (Schrödinger)	Numerical results (Dirac)
10^0	5.736	5.736	5.734
10^1	7.373	7.374	7.371
10^2	9.141	9.142	9.136
10^3	11.00	11.00	10.99
10^4	12.93	12.93	12.91
10^5	14.91	14.91	14.88
10^6	16.93	16.93	16.89
10^7	18.98	18.98	18.92
10^8	21.06	21.06	20.98
10^9	23.16	23.15	23.05
10^{10}	25.27	25.27	25.14

Results for $Z = 1$ with the account of screening

$$\bar{V}(z) = -\frac{Ze^2}{a_H} \left[1 - e^{-\sqrt{6m_e^2}|z|} + e^{-\sqrt{(2/\pi)e^3 B + 6m_e^2}|z|} \right] \sqrt{\frac{\pi}{2}} \exp\left(\frac{z^2}{2a_H^2}\right) \operatorname{erfc}\left(\frac{|z|}{\sqrt{2}a_H}\right)$$

For λ we get:

$\frac{B}{B_0}$	Machet-Vysotsky formula (Schrödinger)	Numerical results (Schrödinger)	Numerical results (Dirac)
10^0	5.736	5.735	5.734
10^1	7.367	7.371	7.368
10^2	9.082	9.095	9.090
10^3	10.53	10.584	10.575
10^4	11.11	11.231	11.219
10^5	11.20	11.357	11.346
10^6	11.21	11.381	11.369
10^7	11.21	11.386	11.375
10^8	11.21	11.387	11.376

Pointlike Coulomb center with charge $Z > 137$ can not exist.

1945-1970

But if the finite size of a nucleus is taken into account then Dirac equation can be solved for $Z > 137$. At $Z = Z_{cr} \approx 170$ ground energy level reaches lower continuum ($\varepsilon = -m_e$) and spontaneous electron-positron pair production occurs. Electrons occupy ground level and one can observe two free positrons.

Critical charge in a magnetic field

V.N. Oraevskii, A.I. Rez, V.B. Semikoz, JETP, Vol. 45, No 3, p. 428 (March 1977)

Solution found for $B \gg m^2 e^3 Z^2$; $B \gg \frac{m^2}{e(Ze^2)^2}$

$$Ze^2 \ln \left(2 \frac{\sqrt{m_e^2 - \varepsilon^2}}{\sqrt{eB}} \right) + \arctan \left(\sqrt{\frac{m_e + \varepsilon}{m_e - \varepsilon}} \right) + \arg \Gamma \left(-\frac{Ze^2 \varepsilon}{\sqrt{m_e^2 - \varepsilon^2}} + iZe^2 \right) \\ - \arg \Gamma(1 + 2iZe^2) - \frac{Ze^2}{2} (\ln 2 + \gamma) = \frac{\pi}{2} + n\pi ,$$

for ground energy level $n = 0$ for $\varepsilon > 0$ and $n = 1$ for $\varepsilon < 0$.

Critical charge:

$$\frac{B}{B_0} = 2(Z_{cr} e^2)^2 \exp \left(-\gamma + \frac{\pi - 2 \arg \Gamma(1 + 2iZ_{cr} e^2)}{Z_{cr} e^2} \right).$$

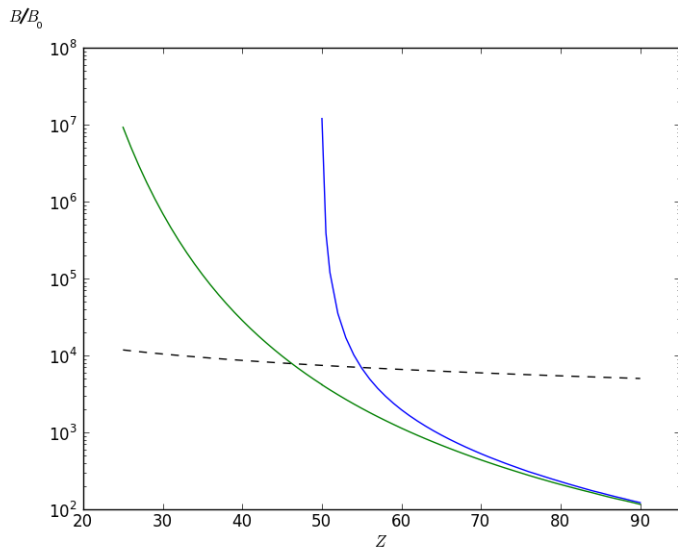
$$B/B_0 \quad 10^2 \quad 10^3 \quad 2 \cdot 10^4$$

$$Z_{cr} \quad 96 \quad 61 \quad 41$$

Results for $Z = 40$

$\frac{B}{B_0}$	ε/m_e (ORS equation)	ε/m_e (Numerical results)	ε/m_e (Numerical results with the account of screening)
10^0	0.819	0.850	0.850
10^1	0.653	0.667	0.667
10^2	0.336	0.339	0.346
10^3	-0.158	-0.159	-0.0765
10^4	-0.758	-0.759	-0.376
$2 \cdot 10^4$	-0.926	-0.927	-0.423
...	at $B/B_0 \approx 2.85 \cdot 10^4$, $\varepsilon = -m_e$...
10^5	—	—	-0.4887
10^6	—	—	-0.5241
10^7	—	—	-0.5351
10^8	—	—	-0.5386
10^9	—	—	-0.5397

Critical nucleus charge with the account of screening



- Analytical formula for the modified potential in the transverse plane was derived;
- There is no large contributions to the potential from higher loops;
- For $Z = 1$ relativistic effects are not important;
- For large Z the effect of screening significantly changes the energy of the ground level;
- Only ions with $Z \gtrsim 52$ can reach criticality.