Third order wave equation in Duffin-Kemmer-Petiau theory. Massive case

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In the paper by M. Nowakowski (1998) devoted to the problem of electromagnetic coupling in the Duffin-Kemmer-Petiau (DKP) theory several rather unusual circumstances relating to a second order DKP equation have been pointed out. One of these is connected with the fact that the second order Kemmer equation hasn't a back-transformation which would allow us to obtain solutions of the first order DKP equation from solutions of the second order equation. The reason of the latter is that the Klein-Gordon-Fock divisor (H. Umezawa and A. Visconti (1956)) in the spin-1 case

$$d(\partial) = \frac{1}{m}(\Box + m^2)I + i\beta_\mu\partial^\mu - \frac{1}{m}\beta_\mu\beta_\nu\partial^\mu\partial^\nu \tag{1}$$

ceases to be commuted with the original DKP operator

$$L(\partial) \equiv i\beta_\mu\partial^\mu - mI, \tag{2}$$

when we introduce the interaction with an external electromagnetic field within the minimal coupling scheme $\partial^\mu \to D^\mu \equiv \partial^\mu + ieA^\mu$, i.e.

$$[d(D), L(D)] \neq 0.$$

Here the matrices β_μ obey the famous trilinear relation:

$$\beta_\mu\beta_\nu\beta_\lambda + \beta_\lambda\beta_\nu\beta_\mu = g_{\mu\nu}\beta_\lambda + g_{\lambda\nu}\beta_\mu.$$

In particular, the lack of commutativity does not give a possibility within the framework of the DKP theory to construct the path integral representation for the Green's function of a spin-1 particle in a background gauge field in a spirit of the approaches developed for a spin-1/2 particle (E.S. Fradkin and D.M. Gitman (1991)).

M. Nowakowski has suggested a way how this problem may be circumvented. To achieve the commutativity of the reciprocal operator $d(D)$ and the DKP operator $L(D)$ in the presence of an external gauge field we have to give up the requirement that the product of these two operators is an operator of the Klein-Gordon-Fock type, i.e.

$$d(D)L(D) \neq -(D^2 + m^2)I + \mathcal{G}[A_\mu],$$

where $\mathcal{G}[A_\mu]$ is a functional of the potential A_μ, which vanishes in the interaction free case. In other words it is necessary to introduce into consideration not the second order, but a higher order wave equation which would have a back-transformation to the solutions of the first order equation.

The purpose of this work is to present a consistent approach of deriving the third order wave equation within the framework of the massive Duffin-Kemmer-Petiau theory in the free and interacting cases. As a basis we take the cubic roots of unity $(q, q^2, 1)$, where the primitive roots q and q^2 are given by the formulae

$$q = \mathrm{e}^{2\pi i/3} = -\frac{1}{2} + i\,\frac{\sqrt{3}}{2}, \qquad q^2 = \mathrm{e}^{4\pi i/3} = -\frac{1}{2} - i\,\frac{\sqrt{3}}{2},$$

and as the spin matrices we take the β-matrices of the DKP algebra. The equation in the following form (R. Kerner (1992)):

$$(i\beta_\mu\partial^\mu - qmI)(i\beta_\nu\partial^\nu - q^2mI)(i\beta_\lambda\partial^\lambda - mI) = -i\,\Box\beta_\mu\partial^\mu - m^3 I \qquad (3)$$

will be analog of the second order Dirac equation. On the right-hand side of (3) we now have the differential operator of the third order, which we take as a “genuine” expression for the third order wave operator in DKP theory. In deriving (3), one of the properties of the roots of unity, namely

$$1 + q + q^2 = 0 \qquad (4)$$

and the operator identity: $\beta_\mu\beta_\nu\beta_\lambda\partial^\mu\partial^\nu\partial^\lambda = \Box\beta_\mu\partial^\mu$ were used.

Cubic root of the Klein-Gordon-Fock equation

In the beginning we consider a question of the construction of cubic root of the second order massive Klein-Gordon-Fock operator. This problem in the general statement has been investigated by M.S. Plyushchay and M. Rausch de Traubenberg (2000). Here, we would like to examine it again and to look how far we can proceed in solving this problem remaining within the framework of DKP formalism only.

We state a question of defining a matrix A such that the following relation holds:

$$\left[A(i\beta_{\mu}\partial^{\mu} - mI)\right]\left[A(i\beta_{\nu}\partial^{\nu} - mI)\right]\left[A(i\beta_{\lambda}\partial^{\lambda} - mI)\right] = -(\Box + m^2)I. \qquad (5)$$

By virtue of the relation

$$-(\Box + m^2)I = d(\partial)\left(i\beta_{\mu}\partial^{\mu} - mI\right),$$

we can examine instead of (5) the following more simple equation:

$$A\left(i\beta_{\mu}\partial^{\mu} - mI\right)A\left(i\beta_{\nu}\partial^{\nu} - mI\right)A = d(\partial). \qquad (6)$$

Here we recall that the operator

$$d(\partial) = mI + i\beta_{\mu}\partial^{\mu} + \left(2g_{\mu\nu} - \{\beta_{\mu}, \beta_{\nu}\}\right)\frac{\partial^{\mu}\partial^{\nu}}{2m}$$

is the Klein-Gordon-Fock divisor in the spin-1 case.

By equating the coefficients of partial derivatives we obtain a system of algebraic equations for the unknown matrix A:

$$A^3 = \frac{1}{m}\, I, \tag{7}$$

$$A\beta_\mu A^2 + A^2\beta_\mu A = -\frac{1}{m}\,\beta_\mu, \tag{8}$$

$$A\beta_\mu A\beta_\nu A + A\beta_\nu A\beta_\mu A = -\frac{1}{m}\left[2g_{\mu\nu} I - \{\beta_\mu, \beta_\nu\}\right]. \tag{9}$$

Let us now introduce an important matrix ω setting (E. Schrödinger (1943))

$$\omega = \frac{i}{4}\,\epsilon^{\mu\nu\lambda\sigma}\beta_\mu\beta_\nu\beta_\lambda\beta_\sigma. \tag{10}$$

We seek the matrix A in the form of the most general expansion in powers of the ω matrix:

$$A = \alpha I + \beta\omega + \gamma\omega^2,$$

where α, β and γ are unknown, generally speaking complex, scalar constants.

The ω-β_μ matrix algebra

In deriving an explicit form of the matrix A we have used the ω-β_μ matrix algebra for the spin-1 case (the matrix ω is identically zero for the spin 0):

$$\omega^3 = \omega,$$

$$\omega^2\beta_\mu + \beta_\mu\omega^2 = \beta_\mu,$$

$$\omega\beta_\mu\omega = 0,$$

$$\beta_\mu\beta_\nu\omega + \omega\beta_\nu\beta_\mu = \omega g_{\mu\nu},$$

$$\omega^2\beta_\mu\beta_\nu = \beta_\mu\beta_\nu\omega^2,$$

$$\beta_\mu\omega\beta_\nu + \beta_\nu\omega\beta_\mu = 0.$$

The next formulae

$$\{\beta_\mu, \beta_\nu\}\omega + \omega\{\beta_\mu, \beta_\nu\} = 2\omega g_{\mu\nu}, \quad (11)$$

$$[\beta_\mu, \beta_\nu]\omega - \omega[\beta_\mu, \beta_\nu] = 0 \quad (12)$$

are an obvious consequence of these relations. The useful contractions are

$$\beta^\mu\omega^2\beta_\mu = 3(1-\omega^2), \quad \beta^\mu\beta^\nu\beta_\mu\beta_\nu = 3-\omega^2, \quad \beta^\mu\beta_\nu\beta_\mu = \beta_\nu.$$

Just before turning to solving the matrix equations (7)−(9), we would like to make a few comments of a general character. The first two equations are universal in determinate sense. The former defines the mass term and the latter enables us to get rid of the term of the first order in the derivatives. The universality of these matrix equations lies in the fact that they must be satisfied in any case irrespectively of that we take as the right part:

$$\text{either} \quad -(\Box + m^2)I \qquad \text{or} \quad -i\,\frac{1}{m}\,\Box\beta_\mu\partial^\mu - m^2 I.$$

Further, the equations (7) and (8) uniquely define the required matrix A:

$$A = \alpha\biggl(I + i\,\frac{\sqrt{3}}{2}\,\omega - \frac{3}{2}\,\omega^2\biggr), \tag{13}$$

where the parameter α satisfies the condition $\alpha^3 = 1/m$. An explicit form of the matrix A and also the equalities (7) and (8) to which it satisfies, are of fundamental importance for further discussion.

The third equation (9) is not already universal and completely depends on the specific choice of the right-hand side in equalities of the (5) type. This equation must be identically satisfied. If not, we come to the contradiction.

The η_μ matrices

Let us analyse the results of the previous consideration from a slightly different point of view. For this purpose we introduce a new set of matrices η_μ that would satisfy the following condition

$$A\eta_\mu = \mathrm{w}\,\eta_\mu A, \tag{14}$$

where w is some complex number. We use the rule of rearrangement (14) to bring the matrix coefficients preceding the partial derivatives in the expression $\big[A(i\eta_\mu\partial^\mu - mI)\big]^3$ into a simple form:

$$\begin{aligned}
(\partial^\mu)^3: &\qquad A\eta_\mu A\eta_\nu A\eta_\lambda = \mathrm{w}^3\,\frac{1}{m}\,\eta_\mu\eta_\nu\eta_\lambda,\\
(\partial^\mu)^2: &\qquad A\eta_\mu A\eta_\nu A + A\eta_\mu A^2\eta_\nu + A^2\eta_\mu A\eta_\nu = \mathrm{w}\,\varepsilon(\mathrm{w})\,\frac{1}{m}\,\eta_\mu\eta_\nu,\\
(\partial^\mu): &\qquad A^3\eta_\mu + A\eta_\mu A^2 + A^2\eta_\mu A = \varepsilon(\mathrm{w})\,\frac{1}{m}\,\eta_\mu,
\end{aligned} \tag{15}$$

where the function $\varepsilon(z) = 1 + z + z^2 \equiv (z-q)(z-q^2)$. If we set the complex number w equal to q (or q^2), then the following equality will be valid:

$$\big[A(i\eta_\mu\partial^\mu - mI)\big]^3 = -i\,\frac{1}{m}\,\eta_\mu\eta_\nu\eta_\lambda\,\partial^\mu\partial^\nu\partial^\lambda - m^2 I. \tag{16}$$

The η_μ matrices

Let us now turn to the construction of an explicit form of the matrices η_μ. To this end, we introduce the following *deformed* commutator $[A, \beta_\mu]_q \equiv A\beta_\mu - q\beta_\mu A$, where the root q is a deformation parameter. We rearrange the matrix A to the left

$$[A, \beta_\mu]_q = A\big(\beta_\mu - mqA^2\beta_\mu A\big) \equiv A\eta_\mu.$$

Here, we have taken into account that $A^{-1} = mA^2$. On the other hand we can rearrange the same matrix to the right

$$[A, \beta_\mu]_q = \big(mA\beta_\mu A^2 - q\beta_\mu\big)A \equiv q^2\eta_\mu A.$$

As the matrix η_μ in (14) it is necessary to take the following expression

$$\eta_\mu = \left(1 + \frac{1}{2}\,q\right)\beta_\mu + \left(\frac{i\sqrt{3}}{2}\right)q\xi_\mu, \quad \xi_\mu = [\omega, \beta_\mu], \tag{17}$$

and the complex parameter w should be set equal to q^2. Thus, the rule of the rearrangement of the matrices A and η_μ can be written in the final form

$$A\eta_\mu = q^2\eta_\mu A. \tag{18}$$

In the choice $\mathrm{w} = q^2$ according to (15) the linear and quadratic in ∂^μ contributions in (16) vanish. However, the contribution in (16) cubic in derivatives after symmetrization with respect to the vector indices also vanishes.

The η_μ matrices

By using an explicit form of the η-matrices, it is not difficult to see that instead of the identity $\beta_\mu\beta_\nu\beta_\lambda\partial^\mu\partial^\nu\partial^\lambda = \Box\beta_\mu\partial^\mu$ we have now

$$\eta_\mu\eta_\nu\eta_\lambda\,\partial^\mu\partial^\nu\partial^\lambda = \lim_{z\to q}\varepsilon(z)\Box\eta_\mu\partial^\mu = 0. \tag{19}$$

Instead of the operator $\left[A(i\eta_\mu\partial^\mu - mI)\right]^3$, we introduce a singular operator:

$$\left[A\left(\frac{i}{\varepsilon^{1/3}(z)}\,\eta_\mu(z)\partial^\mu - mI\right)\right]^3,$$

where now

$$\eta_\mu(z) \equiv \left(1+\frac{1}{2}\,z\right)\beta_\mu + z\left(\frac{i\sqrt{3}}{2}\right)\xi_\mu.$$

In the limit $z\to q$ according to the formulas (15), the contribution, which is linear in ∂^μ behaves as $\varepsilon^{2/3}(z)\to 0$ and the contribution, which is quadratic in ∂^μ behaves as $\varepsilon^{1/3}(z)\to 0$. On the strength of Eq. (19), nonvanishing contribution gives us only the term cubic in ∂^μ and thus we finally obtain the desired expression

$$\lim_{z\to q}\left[A\left(\frac{i}{\varepsilon^{1/3}(z)}\,\eta_\mu(z)\partial^\mu - mI\right)\right]^3 = \left(-i\,\frac{1}{m}\,\Box\eta_\mu\partial^\mu - m^2I\right), \tag{20}$$

where $\lim_{z\to q}\eta_\mu(z) = \eta_\mu(q) \equiv \eta_\mu$.

Properties of the η-matrices

Let us derive a number of relations to which the matrices η_μ satisfy. Our first step is to consider the commutator of two η-matrices. As the commutation relation for the η-matrices we take the following expression:

$$\lim_{z \to q} \frac{1}{\varepsilon(z)}\, i\,[\eta_\mu(z), \eta_\nu(z)] = i\,[\beta_\mu, \beta_\nu]. \tag{21}$$

Let us consider the double commutation relation with the η-matrices. By using (21) we have

$$\lim_{z \to q} \frac{1}{\varepsilon(z)}\, i\,[[\eta_\mu(z), \eta_\nu(z)], \eta_\lambda(z)] = i\,(\eta_\mu g_{\lambda\nu} - \eta_\nu g_{\lambda\mu}). \tag{22}$$

The relation (22) enables us, in particular, to clear up a question about the relativistic invariance of wave equation

$$A\left(\frac{i}{\varepsilon^{1/3}(z)}\,\eta_\mu(z)\,\partial^\mu - m\,I\right)\psi(x;z) = 0. \tag{23}$$

Further, the trilinear relation to which the matrices η_μ have to satisfy (analog of the DKP-relation) takes the following form:

$$\lim_{z \to q} \frac{1}{\varepsilon(z)}\,\bigl(\eta_\mu(z)\eta_\lambda(z)\eta_\nu(z) + \eta_\nu(z)\eta_\lambda(z)\eta_\mu(z)\bigr) = g_{\mu\lambda}\eta_\nu + g_{\nu\lambda}\eta_\mu. \tag{24}$$

The rule of hermitian conjugation is $\boldsymbol{\eta}_0\eta_\mu^\dagger\boldsymbol{\eta}_0 = -q\eta_\mu$, where $\boldsymbol{\eta}_0 = 2\beta_0^2 - 1$.

Interacting case

Let us consider generalization of the result obtained to the case of the interaction with an external electromagnetic field introduced through the minimal coupling scheme: $\partial^\mu \to D^\mu \equiv \partial^\mu + ieA^\mu(x)$. We have for the cube of the linear operator in the presence of the gauge field:

$$\lim_{z\to q}\left[A\left(\frac{i}{\varepsilon^{1/3}(z)}\,\eta_\mu(z)\,D^\mu - mI\right)\right]^3$$
$$= -i\,\frac{1}{m}\,(\eta_\mu D^\mu)\,D^2 - m^2 I$$
$$-\,\frac{5}{6}\left(\frac{ie}{2m}\right)\left(\eta_\mu S^{(\beta)}_{\nu\lambda}\right)D^\mu F^{\nu\lambda} + \frac{1}{3}\left(\frac{ie}{2m}\right)\left(S^{(\beta)}_{\mu\nu}\eta_\lambda - g_{\mu\nu}\eta_\lambda\right)D^\mu F^{\nu\lambda} \tag{25}$$
$$-\,\frac{4}{6}\left(\frac{ie}{2m}\right)\left(\eta_\mu S^{(\beta)}_{\nu\lambda}\right)F^{\nu\lambda}D^\mu + \frac{2}{3}\left(\frac{ie}{2m}\right)\left(S^{(\beta)}_{\mu\nu}\eta_\lambda - g_{\mu\nu}\eta_\lambda\right)F^{\nu\lambda}D^\mu,$$

where we introduced the spin-tensor: $S^{(\beta)}_{\mu\nu} \equiv i\,[\beta_\mu, \beta_\nu]$ and used the identity:

$$\eta_\nu\eta_\mu\eta_\lambda - \eta_\lambda\eta_\mu\eta_\nu = \eta_\mu\,[\eta_\nu, \eta_\lambda] - \big([\eta_\mu, \eta_\nu]\,\eta_\lambda - [\eta_\mu, \eta_\lambda]\,\eta_\nu\big).$$

We performed the detailed comparison of the expression for the third order wave operator with a similar expression earlier obtained by M. Nowakowski.

The general structure of a solution of the first-order differential equation

Let's analyse the general structure of a solution of the equation of the first order in the derivatives

$$\hat{\mathcal{L}}(z, D)\psi(x; z) = 0. \tag{26}$$

Here, we have introduced a short-hand notation for the first-order differential operator

$$\hat{\mathcal{L}}(z, D) \equiv A\left(\frac{i}{\varepsilon^{1/3}(z)}\,\eta_\mu(z)\,D^\mu - mI\right). \tag{27}$$

In the notation of this operator we have explicitly separated out the dependence on the deformation parameter z. A solution of Eq. (26) can be unambiguous presented in the following form:

$$\psi(x; z) = \left[\hat{\mathcal{L}}(z, D)\right]^2 \varphi(x; z), \tag{28}$$

where in turn the function $\varphi(x; z)$ is a solution of the third order wave equation

$$\left[\hat{\mathcal{L}}(z, D)\right]^3 \varphi(x; z) = 0. \tag{29}$$

Let us analyze in a more detail the structure of the solution $\psi(x; z)$ in the form (28). For simplicity we restrict our attention to the interaction free case. We introduce the following notation:

$$\delta \equiv z - q.$$

The general structure of a solution of the first-order differential equation

We can present matrices $\eta_\mu(z)$ in the form of an expansion in terms of δ:

$$\eta_\mu(z) = \left(1 + \frac{1}{2}\,z\right)\beta_\mu + z\left(\frac{i\sqrt{3}}{2}\right)\xi_\mu = \eta_\mu + \delta\,\eta'_\mu. \qquad (30)$$

Here, the matrices η'_μ have the form

$$\eta'_\mu \equiv \frac{d\eta_\mu(z)}{dz}\bigg|_{z=q} = \frac{1}{2}\,\beta_\mu + \left(\frac{i\sqrt{3}}{2}\right)\xi_\mu. \qquad (31)$$

The first order differential operator $\hat{\mathcal{L}}(z,\partial)$ can be rewritten as follows:

$$\hat{\mathcal{L}}(z,\partial) = \left[A\left(\frac{1}{\delta^{1/3}}\,\frac{i}{\varrho^{1/3}}\,\eta_\mu\partial^\mu + \delta^{2/3}\frac{i}{\varrho^{1/3}}\,\eta'_\mu\partial^\mu - mI\right)\right],\ \ \varrho \equiv q - q^2. \qquad (32)$$

A solution of the first order equation can be obtained in the form of formal series in positive and *negative* powers of the parameter $\delta^{1/3}$:

$$\psi(x;z) = \ldots + \frac{1}{\delta}\,\psi_{-1}(x) + \frac{1}{\delta^{2/3}}\,\psi_{-2/3}(x) + \frac{1}{\delta^{1/3}}\,\psi_{-1/3}(x) + \qquad (33)$$

$$+\,\psi_0(x) + \delta^{1/3}\psi_{1/3}(x) + \ldots\,.$$

The general structure of a solution of the first-order differential equation

The solution $\varphi(x;z)$ is regular at $z = q$. It can be presented in the form of a formal series in positive powers of $\delta^{1/3}$:

$$\varphi(x;z) = \varphi_0(x) + \delta^{1/3}\varphi_{1/3}(x) + \delta^{2/3}\varphi_{2/3}(x) + \delta\varphi_1(x) + \dots . \quad (34)$$

Substituting the expansions (33) and (34) into the relation

$$\psi(x;z) = \left[\hat{\mathcal{L}}(z,D)\right]^2 \varphi(x;z)$$

and collecting terms of the same power in $\delta^{1/3}$, we obtain that $\psi_{-1} = \psi_{-4/3} = \dots = 0$ and

$$\psi_{-2/3} = -\frac{1}{\varrho^{2/3}}\left(A\eta_\mu A\eta_\nu\right)\partial^\mu\partial^\nu\varphi_0, \quad (35)$$

$$\psi_{-1/3} = -\frac{i}{\varrho^{1/3}}\, m\left(A\eta_\mu A + A^2\eta_\mu\right)\partial^\mu\varphi_0 - \frac{1}{\varrho^{2/3}}\left(A\eta_\mu A\eta_\nu\right)\partial^\mu\partial^\nu\varphi_{1/3},$$

$$\psi_0 = m^2A^2\varphi_0 - \frac{i}{\varrho^{1/3}}\, m\left(A\eta_\mu A + A^2\eta_\mu\right)\partial^\mu\varphi_{1/3} - \frac{1}{\varrho^{2/3}}\left(A\eta_\mu A\eta_\nu\right)\partial^\mu\partial^\nu\varphi_{2/3}$$

and so on.

The general structure of a solution of the first-order differential equation

The differential equations to which the functions $\varphi_0(x)$, $\varphi_{1/3}(x)$, ... must satisfy, are defined by the corresponding expansion of the cube of the operator $\hat{\mathcal{L}}(z, \partial)$. With allowance for an expansion of $\big[\hat{\mathcal{L}}(z, \partial)\big]^3$, we get

1. the singular contributions:

$$\delta^{-1}: \quad -\frac{i}{\varrho}\left(A\eta_\mu A\eta_\nu A\eta_\lambda\right)\partial^\mu\partial^\nu\partial^\lambda,$$

$$\delta^{-2/3}: \quad \frac{1}{\varrho^{2/3}}\, m\left(A\eta_\mu A^2\eta_\nu + A^2\eta_\mu A\eta_\nu + A\eta_\mu A\eta_\nu A\right)\partial^\mu\partial^\nu,$$

$$\delta^{-1/3}: \quad \frac{i}{\varrho^{1/3}}\, m^2\left(A^3\eta_\mu + A\eta_\mu A^2 + A^2\eta_\mu A\right)\partial^\mu.$$

The first expression vanishes by virtue of nilpotency property: $(\eta\cdot\partial)^3 = 0$. The others two vanish on the strength of the property: $1 + q + q^2 = 0$.

2. the regular contributions:

$$\delta^0: \quad \left(-i\,\frac{1}{m}\,\Box\eta_\mu\partial^\mu - m^2 I\right),$$

$$\delta^{1/3}: \quad -\,\varrho^{1/3}\left[\eta_\mu\eta_\nu - q\,\eta'_\mu\eta_\nu + q^2\eta_\mu\eta'_\nu\right]\partial^\mu\partial^\nu, \; \ldots .$$

Substituting the expansion of the cube of the operator $\hat{\mathcal{L}}(z;\partial)$ and the expansion of $\varphi(x;z)$ in $\big[\hat{\mathcal{L}}(z,D)\big]^3\varphi(x;z)=0$, we obtain the desired equations for the functions $\varphi_0(x)$, $\varphi_{1/3}(x)$, ...

$$\delta^0 \quad : \quad \left(-i\,\frac{1}{m}\,\Box\eta_\mu\partial^\mu - m^2 I\right)\varphi_0(x) = 0, \tag{36}$$

$$\delta^{1/3} \quad : \quad \left(-i\,\frac{1}{m}\,\Box\eta_\mu\partial^\mu - m^2 I\right)\varphi_{1/3}(x) - \tag{37}$$
$$- \,\varrho^{1/3}\big[\eta_\mu\eta_\nu - q\,\eta'_\mu\eta_\nu + q^2\,\eta_\mu\eta'_\nu\big]\,\partial^\mu\partial^\nu\varphi_0(x) = 0$$

and so on. We have used the following relations between η- and η'-matrices:

$$\begin{cases} A^2\eta'_\mu A = \dfrac{1}{m}\,(\eta_\mu + q^2\,\eta'_\mu), \\[2mm] A\eta'_\mu A^2 = \dfrac{1}{m}\,(-\eta_\mu + q\,\eta'_\mu), \end{cases}$$

$$(\eta'_\mu\eta_\lambda\eta_\nu + \eta_\nu\eta_\lambda\eta'_\mu) + (\eta_\mu\eta'_\lambda\eta_\nu + \eta_\nu\eta'_\lambda\eta_\mu) + (\eta_\mu\eta_\lambda\eta'_\nu + \eta'_\nu\eta_\lambda\eta_\mu) =$$
$$= \varrho\,(g_{\mu\lambda}\eta_\nu + g_{\nu\lambda}\eta_\mu).$$

Let the operator $\hat{\mathcal{L}}$ be the cubic root of the third order wave operator. Let us assume that this operator is a para-Fermi operator (parastatistics of order two). In this case the Fock-Schwinger proper-time representation for the inverse operator $\hat{\mathcal{L}}^{-1}$ is

$$\frac{1}{\hat{\mathcal{L}}} \equiv \frac{\hat{\mathcal{L}}^2}{\hat{\mathcal{L}}^3} = i\int_0^\infty d\tau \int \frac{d^2\chi}{\tau^2}\, \mathrm{e}^{-i\tau\left(\hat{H}(z) - i\epsilon\right) + \frac{1}{2}\left(\tau[\chi, \hat{\mathcal{L}}] + \frac{1}{4}\tau^2[\chi, \hat{\mathcal{L}}]^2\right)},$$

$$\epsilon \to +0, \qquad (38)$$

where

$$\hat{H}(z) \equiv \hat{\mathcal{L}}^3(z, D)$$

and χ is a para-Grassmann variable of order $p = 2$ (i.e. $\chi^3 = 0$) with the rules of an integration (Y. Ohnuki, S. Kamefuchi (1980)):

$$\int d^2\chi = \int d^2\chi\,[\chi, \hat{\mathcal{L}}] = 0, \quad \int d^2\chi\,[\chi, \hat{\mathcal{L}}]^2 = 4\hat{\mathcal{L}}^2.$$

As a proper para-supertime here it is necessary to take a triple (τ, χ, χ^2).

The expression (38) can be taken as the starting one for the construction of the desired path integral representation for the spin-1 particle propagator with the use of an appropriate system of coherent states. Here, it would be possible to make good use of the known for a long time connection between the trilinear algebra of β-matrices and the para-Fermi algebra of order two (D.V. Volkov (1959), N.A. Chernikov (1962), C. Ryan (1963)).

However, it should be specially noted that the Fock-Schwinger proper-time representation (38) for an arbitrary value of the deformation parameter z is meant here as a purely formal one, since the fact of the presence of supersymmetry corresponding to parastatistics of order two has not been demonstrated by us explicitly. In addition, the indicated parasupersymmetry most likely does not take place for z distinct from q (or from q^2). The final conclusion about the existence of this symmetry can be made only after constructing the appropriate path integral representation and passage to the limit $z \to q$, in particular

$$\hat{H} = \lim_{z \to q} \hat{H}(z) = \lim_{z \to q} \left[A\left(\frac{i}{\varepsilon^{1/3}(z)}\, \eta_{\mu}(z)\, D^{\mu} - mI \right) \right]^3 .$$

In closing let us remark another fact closely related to the subject matter of this talk. In the literature there are very few papers dealing with the problem of construction of an action for a relativistic classical spin-1 particle using the para-Grassmann variables with subsequent quantization of the classical model (V.D. Gershun and V.I. Tkach (1985), G.P. Korchemsky (1991, 1992), N. Fleury and M. Rausch de Traubenberg (1996)). Here we would like to note only that in the action suggested in these papers there are the linear and quadratic in para-Grassmann variable χ terms, which are similar to terms in the exponential function in our expression (38).

In our case these terms automatically appear in defining Fock-Schwinger proper-time representation, and in the above-cited works they insure the invariance of the action under the local world-line para-SUSY transformation. However, the kinetic part of the action in these works was chosen in a complete analogy with the kinetic part for the classical models of a Dirac particle, whereas we expect based on the general formula (38) that the situation here may be more complicated.

References

1. M. Nowakowski, *Phys. Lett. A* **244** (1998) 329.
2. H. Umezawa and A. Visconti, *Nucl. Phys. B* **1** (1956) 348.
3. E.S. Fradkin and D.M. Gitman, *Phys. Rev. D* **44** (1991) 3230.
4. R. Kerner, *J. Math. Phys.* **33** (1992) 403; *Class. Quant. Grav.* **9** (1992) 137.
5. M.S. Plyushchay, M. Rausch de Traubenberg, *Phys. Lett. B* **477** (2000) 276.
6. Y. Ohnuki and S. Kamefuchi, *J. Math. Phys.* **21** (1980) 609.
7. V.D. Gershun, V.I. Tkach, *Problems of Nuclear Physics and Cosmic Rays* (Kharkov University Press) **23** (1985) 42.
8. G.P. Korchemsky, *Phys. Lett. B* **267** (1991) 497; *Int. J. Mod. Phys. A* **7** (1992) 3493.
9. N. Fleury and M. Rausch de Traubenberg, *Mod. Phys. Lett. A* **11** (1996) 899.
10. Yu.A. Markov, M.A. Markova, A.I. Bondarenko, *Phys. Rev. D* **92** (2015) 105017(24).

Thanks for attention!