Neutrino oscillations in Quantum Mechanics*

- massive neutrinos.

- mixing: fields which enter in weak interaction Lagrangian do not have definite masses $\Leftrightarrow$ mass term not diagonal.

- for two flavors, we have:

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$$

*S.M.Bilenky and B.Pontecorvo, Phys. Rep. (1978)

- Hamiltonian diagonal in the mass eigenstates $\Rightarrow$ time evolution:

$$|\nu_e(t)\rangle = \cos\theta\, e^{-i\omega_1 t}\,|\nu_1\rangle + \sin\theta\, e^{-i\omega_2 t}\,|\nu_2\rangle$$

Flavor oscillations:

$$P_{e\to e}(t) \;=\; |\langle\nu_e|\nu_e(t)\rangle|^2 = 1 - \sin^2 2\theta\ \sin^2\left(\frac{\Delta\omega}{2}t\right)$$

$$P_{e\to \mu}(t) \;=\; |\langle\nu_\mu|\nu_e(t)\rangle|^2 = \sin^2 2\theta\ \sin^2\left(\frac{\Delta\omega}{2}t\right)$$

Flavor conservation:

$$|\langle\nu_e|\nu_e(t)\rangle|^2 + |\langle\nu_\mu|\nu_e(t)\rangle|^2 = 1$$

Consider production process (pion decay):

$$\pi^+ \rightarrow \mu^+ + \nu_\mu$$

and assume that all mass components of ν_μ are created with the same momentum. Then we could naively associate to $|\nu_{\mathbf{k},\mu}\rangle$ an energy:

$$E_\mu(k) \equiv \langle \nu_{\mathbf{k},\mu}|H|\nu_{\mathbf{k},\mu}\rangle = \omega_{k,2}\cos^2\theta + \omega_{k,1}\sin^2\theta$$

with $\omega_{k,i} = \sqrt{k^2 + m_i^2}$ and talk about modified dispersion relations...

- However, this is not correct since the equal momentum assumption can be valid only in a particular Lorentz frame. In general, different massive neutrinos have different energies and momenta*.

- Fundamental entities are the mass eigenstates: In a production process, energy-momentum conservation is assumed for each mass component and either $|\nu_1\rangle$ or $|\nu_2\rangle$ are produced each time.

*C.Giunti, Mod.Phys.Lett.A (2001)

Motivation

- The usual treatment of neutrinos ignores quantum field theory.
- Neutrino oscillation are only calculated for relativistic neutrinos but e. g. early universe neutrinos are non-relativistic.

Flavor mixing in Quantum Field Theory

Consider mixing relations for two Dirac fields

$$\nu_e(x) \;=\; \nu_1(x)\cos\theta + \nu_2(x)\sin\theta$$

$$\nu_\mu(x) \;=\; -\nu_1(x)\sin\theta + \nu_2(x)\cos\theta$$

ν_1, ν_2 are fields with definite masses.

The above mixing transformations connect the two quadratic forms:

$$\mathcal{L} = \bar{\nu}_1\,(i\,\partial\!\!\!/ - m_1)\,\nu_1 + \bar{\nu}_2\,(i\,\partial\!\!\!/ - m_2)\,\nu_2$$

and

$$\mathcal{L} \;=\; \bar{\nu}_e\,(i\,\partial\!\!\!/ - m_e)\,\nu_e + \bar{\nu}_\mu\,(i\,\partial\!\!\!/ - m_\mu)\,\nu_\mu \;-\; m_{e\mu}\,\left(\,\bar{\nu}_e\,\nu_\mu + \bar{\nu}_\mu\,\nu_e\,\right)$$

with $m_e = m_1\cos^2\theta + m_2\sin^2\theta,\; m_\mu = m_1\sin^2\theta + m_2\cos^2\theta,\; m_{e\mu} = (m_2 - m_1)\sin\theta\cos\theta.$

ν_i $(i=1,2)$ are free Dirac field operators:

$$\nu_i(x) = \sum_{\mathbf{k},r} \frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}} [\, u^r_{\mathbf{k},i}(t)\, \alpha^r_{\mathbf{k},i} \;+\; v^r_{-\mathbf{k},i}(t)\, \beta^{r\dagger}_{-\mathbf{k},i} \,]$$

with $u^r_{\mathbf{k},i}(t) \,=\, e^{-i\omega_{k,i}t}\, u^r_{\mathbf{k},i}\,,\;\; v^r_{\mathbf{k},i}(t) \,=\, e^{i\omega_{k,i}t}\, v^r_{\mathbf{k},i}$ and $\omega_{k,i} = \sqrt{k^2+m_i^2}$.

Anticommutation relations:

$$\{\nu^{\alpha}_i(x), \nu^{\beta\dagger}_j(y)\}_{t=t'} = \delta^3(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta}\delta_{ij}$$

$$\{\alpha^r_{\mathbf{k},i}, \alpha^{s\dagger}_{\mathbf{q},j}\} = \delta_{\mathbf{kq}}\delta_{rs}\delta_{ij} \quad ; \quad \{\beta^r_{\mathbf{k},i}, \beta^{s\dagger}_{\mathbf{q},j}\} = \delta_{\mathbf{kq}}\delta_{rs}\delta_{ij}$$

Orthonormality and completeness relations:

$$u^{r\dagger}_{\mathbf{k},i} u^s_{\mathbf{k},i} = v^{r\dagger}_{\mathbf{k},i} v^s_{\mathbf{k},i} = \delta_{rs} \quad , \quad u^{r\dagger}_{\mathbf{k},i} v^s_{-\mathbf{k},i} = 0 \quad , \quad \sum_r (u^{r\alpha*}_{\mathbf{k},i} u^{r\beta}_{\mathbf{k},i} + v^{r\alpha*}_{-\mathbf{k},i} v^{r\beta}_{-\mathbf{k},i}) = \delta_{\alpha\beta}\,.$$

Fock space for ν_1, ν_2:

$$\mathcal{H}_{1,2} = \left\{ \alpha^\dagger_{1,2}\,,\; \beta^\dagger_{1,2}\,,\; |0\rangle_{1,2} \right\}.$$

Generator of mixing transformations

Mixing relations can be written as*

$$\nu_e^\alpha(x) \;=\; G_\theta^{-1}(t)\,\nu_1^\alpha(x)\,G_\theta(t)$$

$$\nu_\mu^\alpha(x) \;=\; G_\theta^{-1}(t)\,\nu_2^\alpha(x)\,G_\theta(t)$$

with generator given by:

$$G_\theta(t) = \exp[\theta\left(S_+(t) - S_-(t)\right)]$$

$$S_+(t) \equiv \int d^3\mathbf{x}\,\nu_1^\dagger(x)\nu_2(x) \quad , \quad S_-(t) \equiv \int d^3\mathbf{x}\,\nu_2^\dagger(x)\nu_1(x)$$

– similar results for bosons†

*M.Blasone and G.Vitiello, Annals Phys. (1995)

†M. Blasone, A. Capolupo, O. Romei and G. Vitiello, Phys. Rev. D (2001).

The mixing of the creation and annihilation operators I

- The field operators can be expanded in terms of creation and annihilation operators

$$\hat{\nu}_i(x) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{p},r} \left(a^r_{\boldsymbol{p},i}(t) u^r_{\boldsymbol{p},i} + b^{r\dagger}_{-\boldsymbol{p},i}(t) v^r_{-\boldsymbol{p},i} \right) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}$$

$$\hat{\nu}_\sigma(x) = \frac{1}{\sqrt{V}} \sum_{\boldsymbol{p},r} \left(a^r_{\boldsymbol{p},\sigma}(t) u^r_{\boldsymbol{p},\sigma} + b^{r\dagger}_{-\boldsymbol{p},\sigma}(t) v^r_{-\boldsymbol{p},\sigma} \right) \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}}$$

- The operators are mixed via

$$\begin{pmatrix} a^r_{\boldsymbol{p},e}(t) \\ a^r_{\boldsymbol{p},\mu}(t) \\ b^{r\dagger}_{-\boldsymbol{p},e}(t) \\ b^r_{-\boldsymbol{p},\mu}(t) \end{pmatrix} = \hat{\mathrm{G}}^{-1}(\theta,t) \begin{pmatrix} a^r_{\boldsymbol{p},1}(t) \\ a^r_{\boldsymbol{p},2}(t) \\ b^{r\dagger}_{-\boldsymbol{p},1}(t) \\ b^{r\dagger}_{-\boldsymbol{p},2}(t) \end{pmatrix} \hat{\mathrm{G}}(\theta,t)$$

The Fock space for flavor neutrinos

- The relation

$$0 = {}_{1,2}\langle 0 | a^r_{\boldsymbol{p},i}(t) | 0 \rangle_{1,2} = {}_{1,2}\langle 0 | \hat{\mathrm{G}}(\theta,t) a^r_{\boldsymbol{p},\sigma}(t) \hat{\mathrm{G}}^{-1}(\theta,t) | 0 \rangle_{1,2}$$

allows the definition of a vacuum for the flavor neutrinos

$$|0(\theta,t)\rangle_{e,\mu} = \hat{\mathrm{G}}^{-1}(\theta,t) |0\rangle_{1,2}.$$

- This is similarly possible for all Fock-space state

$$|\nu_e(\theta,t)\rangle_{e,\mu} = \hat{\mathrm{G}}^{-1}(\theta,t) |\nu_1\rangle_{1,2}.$$

The vacuum $|0\rangle_{1,2}$ is not invariant under the action of the generator $G_\theta(t)$:

$$|0(t)\rangle_{e,\mu} \equiv G_\theta^{-1}(t)\ |0\rangle_{1,2} = e^{-\theta\big(S_+(t)-S_-(t)\big)}\ |0\rangle_{1,2}$$

The vacuum $|0(t)\rangle_{e,\mu}$ is a $SU(2)$ generalized coherent state.*

Relation between $|0\rangle_{1,2}$ and $|0(t)\rangle_{e,\mu}$: orthogonality! (for $V \to \infty$)

$$_{1,2}\langle 0|0(t)\rangle_{e,\mu} = \prod_{\mathbf{k}} \Big(1 - \sin^2\theta\ |V_{\mathbf{k}}|^2\Big)^2$$

$$\lim_{V\to\infty}\ _{1,2}\langle 0|0(t)\rangle_{e,\mu} = \lim_{V\to\infty}\ e^{V\int \frac{d^3\mathbf{k}}{(2\pi)^3}\ \ln\left(1-\sin^2\theta\,|V_{\mathbf{k}}|^2\right)^2} = 0$$

with

$$|V_{\mathbf{k}}|^2 \equiv \sum_{r,s} |\ v^{r\dagger}_{-\mathbf{k},1} u^{s}_{\mathbf{k},2}\ |^2\ , \qquad 0 \le |V_{\mathbf{k}}|^2 \le \frac{1}{2}$$

*A. Perelomov, *Generalized Coherent States and Their Applications*, (Springer-Verlag, Berlin, 1986)

Properties of the Fock space

- The flavor vacuum is a condensate of mass neutrinos

$$|0(\theta, t)\rangle_{e,\mu} = \hat{\mathrm{G}}^{-1}(\theta, t)|0\rangle_{1,2}.$$

- The infinite space limit leads to a separation of the flavor and mass Fock spaces

$$\lim_{V\to\infty} {}_{1,2}\langle 0|0(\theta, t)\rangle_{e,\mu} = \lim_{V\to\infty} {}_{1,2}\langle 0|\hat{\mathrm{G}}^{-1}(\theta, t)|0\rangle_{1,2} = 0$$

$\Rightarrow$ The Fock spaces for flavor and mass neutrinos are unitarily inequivalent.

$\Rightarrow$ $\hat{\mathrm{G}}(\theta, t)$ is an improper unitary operator.

Condensate structure of $|0\rangle_{e,\mu}$ (use $\epsilon^r = (-1)^r$)

$$|0\rangle_{e,\mu} = \prod_{\mathbf{k},r} \Big[(1 - \sin^2\theta\, |V_{\mathbf{k}}|^2) - \epsilon^r \sin\theta\, \cos\theta\, |V_{\mathbf{k}}|\, (\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2} + \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1})$$

$$+ \epsilon^r \sin^2\theta\, |V_{\mathbf{k}}||U_{\mathbf{k}}|\, (\alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},1} - \alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},2}) + \sin^2\theta\, |V_{\mathbf{k}}|^2\, \alpha^{r\dagger}_{\mathbf{k},1}\beta^{r\dagger}_{-\mathbf{k},2}\alpha^{r\dagger}_{\mathbf{k},2}\beta^{r\dagger}_{-\mathbf{k},1} \Big] |0\rangle_{1,2}$$

- 4 kinds of particle-antiparticle pairs with zero momentum and spin.

- Condensation density:

$${}_{e,\mu}\langle 0(t)|\alpha^{r\dagger}_{\mathbf{k},1}\alpha^{r}_{\mathbf{k},1}|0(t)\rangle_{e,\mu} = \sin^2\theta\, |V_{\mathbf{k}}|^2$$

vanishing for $m_1 = m_2$ and/or $\theta = 0$ (in both cases no mixing). Same result for α_2, β_1, β_2.

Structure of the annihilation operators for $|0(t)\rangle_{e,\mu}$:

$$\alpha^r_{\mathbf{k},e}(t) = \cos\theta\,\alpha^r_{\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},2} + \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},2}\right)$$

$$\alpha^r_{\mathbf{k},\mu}(t) = \cos\theta\,\alpha^r_{\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\alpha^r_{\mathbf{k},1} - \epsilon^r V_{\mathbf{k}}(t)\,\beta^{r\dagger}_{-\mathbf{k},1}\right)$$

$$\beta^r_{-\mathbf{k},e}(t) = \cos\theta\,\beta^r_{-\mathbf{k},1} + \sin\theta\left(U^*_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},2} - \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},2}\right)$$

$$\beta^r_{-\mathbf{k},\mu}(t) = \cos\theta\,\beta^r_{-\mathbf{k},2} - \sin\theta\left(U_{\mathbf{k}}(t)\,\beta^r_{-\mathbf{k},1} + \epsilon^r V_{\mathbf{k}}(t)\,\alpha^{r\dagger}_{\mathbf{k},1}\right)$$

Mixing transformation = Rotation $(\cos\theta\,,\,\sin\theta)$ + Bogoliubov transformation $(U_{\mathbf{k}}\,,\,V_{\mathbf{k}})$.

Bogoliubov coefficients:

$$U_{\mathbf{k}}(t) \;=\; u^{r\dagger}_{\mathbf{k},2}u^r_{\mathbf{k},1}\,e^{i(\omega_{k,2}-\omega_{k,1})t} \qquad ; \qquad V_{\mathbf{k}}(t) = \epsilon^r\,u^{r\dagger}_{\mathbf{k},1}v^r_{-\mathbf{k},2}\,e^{i(\omega_{k,2}+\omega_{k,1})t}$$

$$|U_{\mathbf{k}}|^2 + |V_{\mathbf{k}}|^2 = 1$$

The mixing of the creation and annihilation operators II

- The mixing can be written as the matrix

$$\begin{pmatrix} a^r_{\boldsymbol{p},e}(t) \\ a^r_{\boldsymbol{p},\mu}(t) \\ b^{r\dagger}_{-\boldsymbol{p},e}(t) \\ b^r_{-\boldsymbol{p},\mu}(t) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta\, U_{\boldsymbol{p}} & 0 & (-1)^r \sin\theta\, V_{\boldsymbol{p}} \\ -\sin\theta\, U_{\boldsymbol{p}} & \cos\theta & (-1)^r \sin\theta\, V_{\boldsymbol{p}} & 0 \\ 0 & -(-1)^r \sin\theta\, V_{\boldsymbol{p}} & \cos\theta & \sin\theta\, U_{\boldsymbol{p}} \\ -(-1)^r \sin\theta\, V_{\boldsymbol{p}} & 0 & -\sin\theta\, U_{\boldsymbol{p}} & \cos\theta \end{pmatrix} \begin{pmatrix} a^r_{\boldsymbol{p},1}(t) \\ a^r_{\boldsymbol{p},2}(t) \\ b^{r\dagger}_{-\boldsymbol{p},1}(t) \\ b^{r\dagger}_{-\boldsymbol{p},2}(t) \end{pmatrix}$$

$$\text{with} \quad U_p^2 + V_p^2 = 1$$

- For large momenta the matrix can be simplified

$$\boldsymbol{p} \gg \sqrt{m_1 m_2} \qquad U_p \to 1,\, V_p \to 0$$

$\Rightarrow$ For realistic experiments the matrix reduces to the standard one

$$U_{\mathbf{k}}(t) = u^{r\dagger}_{\mathbf{k},2}(t) u^{r}_{\mathbf{k},1}(t) = |U_{\mathbf{k}}| e^{i(\omega_2 - \omega_1)t},$$

$$|U_{\mathbf{k}}| = \left(\frac{\omega_1 + m_1}{2\omega_1}\right)^{\frac{1}{2}} \left(\frac{\omega_2 + m_2}{2\omega_2}\right)^{\frac{1}{2}} \left(1 + \frac{|\mathbf{k}|^2}{(\omega_1 + m_1)(\omega_1 + m_2)}\right).$$

$$V_{\mathbf{k}}(t) = (-1)^r u^{r\dagger}_{\mathbf{k},1}(t) v^{r}_{-\mathbf{k},2}(t) = |V_{\mathbf{k}}| e^{i(\omega_2 + \omega_1)t},$$

$$|V_{\mathbf{k}}| = \left(\frac{\omega_1 + m_1}{2\omega_1}\right)^{\frac{1}{2}} \left(\frac{\omega_2 + m_2}{2\omega_2}\right)^{\frac{1}{2}} \left(\frac{|\mathbf{k}|}{\omega_2 + m_2} - \frac{|\mathbf{k}|}{\omega_1 + m_1}\right).$$

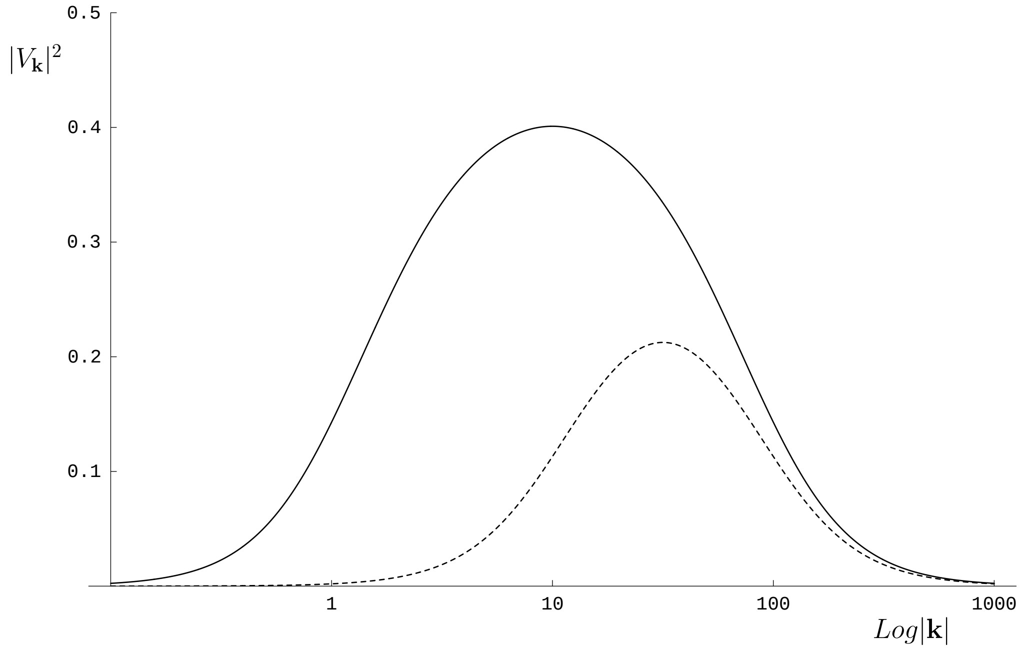
– The term with the Bogoliubov coefficient $|V_{\mathbf{k}}|^2$ is absent in the usual Pontecorvo case.

– The maximum of the function $|V_{\mathbf{k}}|^2$ occurs for $k_{max} = \sqrt{m_1 m_2}$:

$$|V_{k_{max}}|^2 = \frac{1}{2} - \frac{1}{\sqrt{(1+\frac{m_1}{m_2})(1+\frac{m_2}{m_1})}}$$

The condition $|V_{\mathbf{k}}|^2 \ll \frac{1}{2}$ can be then realized and the Pontecorvo dispersion relations taken to be approximately valid.

Condensation density for mixed fermions



Solid line: $m_1 = 1$, $m_2 = 100$; Dashed line: $m_1 = 10$, $m_2 = 100$.

- $V_{\mathbf{k}} = 0$ when $m_1 = m_2$ and/or $\theta = 0$.

- Max. at $k = \sqrt{m_1 m_2}$ with $V_{max} \to \frac{1}{2}$ for $\frac{(m_2 - m_1)^2}{m_1 m_2} \to \infty$.

- $|V_{\mathbf{k}}|^2 \simeq \frac{(m_2 - m_1)^2}{4k^2}$ for $k \gg \sqrt{m_1 m_2}$.

Currents and charges for mixed fermions *

Let us consider the following Lagrangian

$$\mathcal{L} = \bar{\Psi}_m \left(i \not\partial - M_d \right) \Psi_m$$

where $\Psi_m^T = (\nu_1, \nu_2)$ and $M_d = diag(m_1, m_2)$.

- $\mathcal{L}$ is invariant under global $U(1)$ with conserved (Noether) charge Q= total charge.

Consider now the $SU(2)$ transformation:

$$\Psi'_m = e^{i\alpha_j \tau_j} \Psi_m \qquad\qquad j = 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices.

*M.Blasone, P.Jizba and G.Vitiello, Phys. Lett. B (2001)

The associated currents are:

$$\delta\mathcal{L} = i\alpha_j\,\bar{\Psi}_m\,[\tau_j, M_d]\,\Psi_m = -\alpha_j\,\partial_\mu J^\mu_{m,j}$$

$$J^\mu_{m,j} = \bar{\Psi}_m\,\gamma^\mu\,\tau_j\,\Psi_m$$

– The charges $Q_{m,j}(t) \equiv \int d^3x\,J^0_{m,j}(x)$, $j = 1,2,3$, satisfy the $su(2)$ algebra (at equal times): $[Q_{m,j}(t), Q_{m,k}(t)] = i\,\epsilon_{jkl}\,Q_{m,l}(t)\,.$

– The Casimir operator is proportional to the total charge: $C_m = \frac{1}{2}Q$.

- $Q_{m,3}$ is conserved $\Rightarrow$ charge conserved separately for ν_1 and ν_2:

$$Q_1 = \frac{1}{2}Q + Q_{m,3}$$

$$Q_2 = \frac{1}{2}Q - Q_{m,3}$$

– $2Q_{m,2}(t)$ is the generator of the mixing transformations.

The currents in the flavor basis

Let us consider the Lagrangian written in the flavor basis

$$\mathcal{L} = \bar{\Psi}_f \, (i \, \partial\!\!\!/ - M) \, \Psi_f$$

where $\Psi_f^T = (\nu_e, \nu_\mu)$ and $M = \begin{pmatrix} m_e & m_{e\mu} \\ m_{e\mu} & m_\mu \end{pmatrix}$.

Consider the $SU(2)$ transformation:

$$\Psi'_f = e^{i\alpha_j \tau_j} \, \Psi_f \qquad\qquad j = 1, 2, 3.$$

with $\tau_j = \sigma_j/2$ and σ_j being the Pauli matrices.

We have

$$\begin{aligned} \delta\mathcal{L} &= i\alpha_j \, \bar{\Psi}_f \, [\tau_j, M] \, \Psi_f = -\alpha_j \, \partial_\mu J^\mu_{f,j} \, , \\ J^\mu_{f,j} &= \bar{\Psi}_f \, \gamma^\mu \, \tau_j \, \Psi_f \end{aligned}$$

– The charges $Q_{f,j} \equiv \int d^3x\, J^0_{f,j}$, $j = 1, 2, 3$, close the $su(2)$ algebra.

- $Q_{f,3}$ is not conserved $\Rightarrow$ exchange of charge between ν_e and ν_μ.

Define the flavor charges as:

$$
\begin{aligned}
Q_e(t) &\equiv \frac{1}{2}Q + Q_{f,3}(t) \\
Q_\mu(t) &\equiv \frac{1}{2}Q - Q_{f,3}(t)
\end{aligned}
$$

where $Q_e(t) + Q_\mu(t) = Q$.

- The flavor charges are related in a simple way to the Noether charges Q_i:

$$
Q_\sigma(t) = G^{-1}_\theta(t)\, Q_i\, G_\theta(t)
$$

with $(\sigma, i) = (e, 1), (\mu, 2)$.

The flavor charges take the form ($\sigma = e, \mu$):

$$Q_\sigma(t) \;=\; \int d^3\mathbf{x}\, \nu_\sigma^\dagger(x)\, \nu_\sigma(x) = \sum_{\mathbf{k},r} \Big(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t) \alpha^{r}_{\mathbf{k},\sigma}(t) \;-\; \beta^{r\dagger}_{-\mathbf{k},\sigma}(t) \beta^{r}_{-\mathbf{k},\sigma}(t) \Big) \,.$$

• Electron neutrino state with definite momentum and helicity:

$$|\nu^r_{\mathbf{k},e}\rangle \;\equiv\; \alpha^{r\dagger}_{\mathbf{k},e}(0)\, |0\rangle_{e,\mu}$$

$$= \Big[\cos\theta\, \alpha^{r\dagger}_{\mathbf{k},1} + |U_\mathbf{k}|\ \sin\theta\, \alpha^{r\dagger}_{\mathbf{k},2} - \epsilon\, |V_\mathbf{k}|\ \sin\theta\, \alpha^{r\dagger}_{\mathbf{k},1} \alpha^{r\dagger}_{\mathbf{k},2} \beta^{r\dagger}_{\mathbf{k},1} \Big] \prod_{\mathbf{p}\neq\mathbf{k}} G^{-1}_\mathbf{p}(\theta, t)\, |0\rangle_{1,2} \,.$$

We then have:

$${}_{e,\mu}\langle 0 | Q_\sigma(t) |\, 0\rangle_{e,\mu} \;=\; 0$$

$$\langle \nu^r_{\mathbf{k},e} |\, Q_\sigma(t)\, | \nu^r_{\mathbf{k},e}\rangle \;=\; \Big| \Big\{ \alpha^{r}_{\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0) \Big\} \Big|^2 \;+\; \Big| \Big\{ \beta^{r\dagger}_{-\mathbf{k},\sigma}(t), \alpha^{r\dagger}_{\mathbf{k},e}(0) \Big\} \Big|^2$$

Neutrino oscillations I

- The neutrino oscillations can be described by the flavor charges

$$\hat{\mathrm{Q}}_e(t) := \int \mathrm{d}^3x\, \hat{\nu}_e^\dagger(x)\hat{\nu}_e(x) \qquad \hat{\mathrm{Q}}_\mu(t) := \int \mathrm{d}^3x\, \hat{\nu}_\mu^\dagger(x)\hat{\nu}_\mu(x).$$

- The expectation values of the charges yield the probabilities

$$\begin{aligned}
\mathcal{P}_{\nu_e\to\nu_e} &:= {}_{e,\mu}\big\langle \nu_e(\theta,0)\big|\hat{\mathrm{Q}}_e(t)\big|\nu_e(\theta,0)\big\rangle_{e,\mu} \\
&= \left|\left\{a^r_{\boldsymbol{p},e}(t), a^{r\dagger}_{\boldsymbol{p},e}(0)\right\}\right|^2 + \left|\left\{b^{r\dagger}_{-\boldsymbol{p},e}(t), a^{r\dagger}_{\boldsymbol{p},e}(0)\right\}\right|^2 \\
\mathcal{P}_{\nu_e\to\nu_\mu} &:= {}_{e,\mu}\big\langle \nu_e(\theta,0)\big|\hat{\mathrm{Q}}_\mu(t)\big|\nu_e(\theta,0)\big\rangle_{e,\mu} \\
&= \left|\left\{a^r_{\boldsymbol{p},\mu}(t), a^{r\dagger}_{\boldsymbol{p},e}(0)\right\}\right|^2 + \left|\left\{b^{r\dagger}_{-\boldsymbol{p},\mu}(t), a^{r\dagger}_{\boldsymbol{p},e}(0)\right\}\right|^2.
\end{aligned}$$

- Neutrino oscillation formula (exact result)*:

$$\langle \nu^r_{\mathbf{k},e}|\, Q_e(t)\, |\nu^r_{\mathbf{k},e}\rangle \;=\; 1 - |U_{\mathbf{k}}|^2 \sin^2(2\theta)\, \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}\, t\right) - |V_{\mathbf{k}}|^2 \sin^2(2\theta)\, \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}\, t\right)$$

$$\langle \nu^r_{\mathbf{k},e}|\, Q_\mu(t)\, |\nu^r_{\mathbf{k},e}\rangle \;=\; |U_{\mathbf{k}}|^2 \sin^2(2\theta)\, \sin^2\left(\frac{\omega_{k,2}-\omega_{k,1}}{2}\, t\right) + |V_{\mathbf{k}}|^2 \sin^2(2\theta)\, \sin^2\left(\frac{\omega_{k,2}+\omega_{k,1}}{2}\, t\right)$$

– Pontecorvo formula approximately reobtained for $k \gg \sqrt{m_1 m_2}$.

- Space-dependent oscillation formula from flavor current†.

*M.Blasone, P.Henning and G.Vitiello, Phys. Lett. **B** (1999).

†M.Blasone, P. Pires Pachêco and H. Wan Chan Tseung, Phys. Rev. **D**, (2003).

Neutrino oscillations II

- The final result:

$$\mathcal{P}_{\nu_e \to \nu_e} = 1 - \sin^2(2\theta)\left[U_p^2 \sin^2\left(\frac{E_2 - E_1}{2} t\right) + V_p^2 \sin^2\left(\frac{E_2 + E_1}{2} t\right)\right]$$

$$\mathcal{P}_{\nu_e \to \nu_\mu} = \sin^2(2\theta)\left[U_p^2 \sin^2\left(\frac{E_2 - E_1}{2} t\right) + V_p^2 \sin^2\left(\frac{E_2 + E_1}{2} t\right)\right]$$

- For large momenta

$$\boldsymbol{p} \gg \sqrt{m_1 m_2} \qquad U_p \to 1,\, V_p \to 0$$

$\Rightarrow$ For realistic experiments the probabilities reduce to the standard ones

- Flavor states above defined are eigenstates of flavor charges:

$$|\nu^r_{\mathbf{k},\sigma}\rangle \equiv \alpha^{r\dagger}_{\mathbf{k},\sigma}(0)\,|0\rangle_{e,\mu} \qquad ; \qquad |\bar{\nu}^r_{\mathbf{k},\sigma}\rangle \equiv \beta^{r\dagger}_{\mathbf{k},\sigma}(0)\,|0\rangle_{e,\mu}$$

$$\mathbf{Q}_\sigma(0)\,|\nu^r_{\mathbf{k},\sigma}\rangle = +\,|\nu^r_{\mathbf{k},\sigma}\rangle \qquad ; \qquad \mathbf{Q}_\sigma(0)\,|\bar{\nu}^r_{\mathbf{k},\sigma}\rangle = -\,|\bar{\nu}^r_{\mathbf{k},\sigma}\rangle, \qquad \sigma = e, \mu\,.$$

- This is *not* true for QM (Pontecorvo) states:

$$|\nu_e\rangle_P \;=\; \cos\theta\;|\nu_1\rangle + \sin\theta\;|\nu_2\rangle$$

which thus are not properly defined as states with definite flavor:

$${}_P\langle\nu_e|Q_e|\nu_e\rangle_P \neq 1$$

Summary

- We can define a Fock space for flavor neutrinos, which is
  - time-dependent
  - unitarily inequivalent to the Fock space for mass neutrinos
- The oscillation formula can be derived and
  - includes an additional term
  - for $p \gg \sqrt{m_1 m_2}$ recovers the standard formula

For further reading

- M. Blasone and G. Vitiello: hep-ph/9501263
- M. Blasone, P.A. Henning and G. Vietiello: hep-th/9803157
- M. Blasone, P. Jizba and G. Vitiello: hep-th/0103087

Consider now the momentum operator for the definite mass fields:

$$\begin{aligned}\mathbf{P}_j &= -i\int d^3\mathbf{x}\,\psi_j^\dagger(x)\,\nabla\psi_j(x)\\ &= \sum_r\int d^3\mathbf{k}\;\mathbf{k}\left(\alpha^{r\dagger}_{\mathbf{k},j}\alpha^r_{\mathbf{k},j}+\beta^{r\dagger}_{\mathbf{k},j}\beta^r_{\mathbf{k},j}\right)\quad,\quad j=1,2.\end{aligned}$$

and for the flavor fields:

$$\begin{aligned}\mathbf{P}_\sigma(t) &= -i\int d^3\mathbf{x}\,\psi_\sigma^\dagger(x)\,\nabla\psi_\sigma(x)\\ &= \sum_r\int d^3\mathbf{k}\;\mathbf{k}\left(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\alpha^r_{\mathbf{k},\sigma}(t)+\beta^{r\dagger}_{\mathbf{k},\sigma}(t)\beta^r_{\mathbf{k},\sigma}(t)\right)\quad,\quad \sigma=e,\mu\end{aligned}$$

We have:

$$\mathbf{P}_\sigma(t)=G_\theta^{-1}(t)\mathbf{P}_j G_\theta(t)\quad,\quad(\sigma,j)=(e,1),(\mu,2).$$

$$\sum_\sigma\mathbf{P}_\sigma(t)=\sum_j\mathbf{P}_j=\mathbf{P}\quad;\quad[\mathbf{P},G_\theta(t)]=0\quad;\quad[\mathbf{P},H]=0$$

- Flavor states are eigenstates of the momentum operator:

$$\mathbf{P}_\sigma(0)\,|\nu^r_{\mathbf{k},\sigma}\rangle=\mathbf{k}\,|\nu^r_{\mathbf{k},\sigma}\rangle$$

Consider then the Hamiltonian for the definite mass fields:

$$
\begin{aligned}
H_j &= i \int d^3\mathbf{x}\, \psi_j^\dagger(x)\, \partial_0 \psi_j(x) \\
&= \sum_r \int d^3\mathbf{k}\, \left(\alpha^{r\dagger}_{\mathbf{k},j}(t)\, \partial_0\, \alpha^r_{\mathbf{k},j}(t) + \beta^r_{\mathbf{k},j}(t)\, \partial_0\, \beta^{r\dagger}_{\mathbf{k},j}(t) \right) \\
&= \sum_r \int d^3\mathbf{k}\, \omega_{k,j} \left(\alpha^{r\dagger}_{\mathbf{k},j}\, \alpha^r_{\mathbf{k},j} - \beta^r_{\mathbf{k},j}\, \beta^{r\dagger}_{\mathbf{k},j} \right) \quad , \quad j = 1, 2.
\end{aligned}
$$

and for the flavor fields:

$$
\begin{aligned}
H_\sigma(t) &= i \int d^3\mathbf{x}\, \psi_\sigma^\dagger(x)\, \partial_0 \psi_\sigma(x) \\
&= \sum_r \int d^3\mathbf{k}\, \left(\alpha^{r\dagger}_{\mathbf{k},\sigma}(t)\, \partial_0\, \alpha^r_{\mathbf{k},\sigma}(t) + \beta^r_{\mathbf{k},\sigma}(t)\, \partial_0\, \beta^{r\dagger}_{\mathbf{k},\sigma}(t) \right) \quad , \quad \sigma = e, \mu
\end{aligned}
$$

We have:

$$
H_\sigma(t) \neq G_\theta^{-1}(t) H_j G_\theta(t) \quad , \quad (\sigma, j) = (e, 1), (\mu, 2).
$$

$$
\sum_\sigma H_\sigma(t) = \sum_j H_j = H \quad ; \quad [H, G_\theta(t)] \neq 0
$$

Non-trivial result since H and G_θ do not commute.