Lagrangian description of color spinning particle in external fermion field

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Introduction

The formulation of Lagrangian and Hamiltonian descriptions of the dynamics of (pseudo)classical color particles interacting with a background Yang-Mills field has been suggested more than 30 years ago in two fundamental papers by Barducci *et al.* (1977) and Balachandran *et al.* (1977).

However, in some problems we should consider the presence of a background 'non-Abelian' fermion field $\Psi^i_\alpha(x)$ in the system along with a background non-Abelian gauge field $A^a_\mu(x)$.

Introduction

In order to construct the Lagrange description of dynamics for a color spinning particle in external fields one need to have the following objects:

- An interaction Lagrangian the particle with external background fields $A_\mu(x)^a$ and $\Psi^i_\alpha(x)$;
- The field equations – Yang-Mills and Dirac equations for the case of presence of external fermion field.

Interaction Lagrangian

$$L = L_0 + L_m + L_\theta + L_\vartheta + L_\Psi,$$

$$L_0 = -\frac{1}{2e}\,\dot{x}_\mu \dot{x}^\mu + \frac{1}{2i}\left(\frac{d\bar{\psi}}{d\tau}\,\psi - \bar{\psi}\,\frac{d\psi}{d\tau}\right), \quad L_m = -\frac{e}{2}\,m^2,$$

$$L_\theta = i(\theta^{\dagger i} D^{ij} \theta^j) - e\,\frac{g}{4}\,Q^a F^a_{\mu\nu}(\bar{\psi}\sigma^{\mu\nu}\psi),$$

$$L_\vartheta = \frac{i}{2}\,\vartheta^a D^{ab} \vartheta^b - e\,\frac{g}{4}\,\mathcal{Q}^a F^a_{\mu\nu}(\bar{\psi}\sigma^{\mu\nu}\psi),$$

$$L_\Psi = -\frac{e}{\sqrt{2}}\,g\left\{\theta^{\dagger i}(\bar{\psi}_\alpha \Psi^i_\alpha) + (\bar{\Psi}^i_\alpha \psi_\alpha)\theta^i\right\} +$$

$$+\left[\frac{e}{\sqrt{2}}\,g\left(\frac{C_F}{2T_F}\right)Q^a\left\{\theta^{\dagger j}(t^a)^{ji}(\bar{\psi}_\alpha \Psi^i_\alpha) + (\bar{\Psi}^i_\alpha \psi_\alpha)(t^a)^{ij}\theta^j\right\} + (Q^a \rightarrow \mathcal{Q}^a)\right]$$

$$+\left[\frac{e}{\sqrt{2}}\,g\left(\frac{C_F}{2T_F}\right)Q^a F^a_{\mu\nu}\left\{\theta^{\dagger i}(\bar{\psi}\sigma^{\mu\nu}\Psi^i) + (\bar{\Psi}^i \sigma^{\mu\nu}\psi)\theta^i\right\} + (Q^a \rightarrow \mathcal{Q}^a)\right]$$

$$+\frac{e}{\sqrt{2}}\,g\left(\frac{C_F}{2T_F}\right)F^a_{\mu\nu}\left\{\theta^{\dagger j}(t^a)^{ji}(\bar{\psi}\sigma^{\mu\nu}\Psi^j) + (\bar{\Psi}^j \sigma^{\mu\nu}\psi)(t^a)^{ij}\theta^j\right\} + \dots .$$

Spin dynamics in terms of ξ_μ and ξ_5

An alternative approach most generally employed for a description of spin for a massive particle is connected with introduction into consideration of the pseudovector and pseudoscalar dynamical variables $\xi_\mu, \mu = 1, \ldots, 4$, and ξ_5 that are elements of the Grassmann algebra. For these variables an appropriate Lagrangian of the first order time derivative is

$$L = L_0 + L_m + L_\theta,$$

where

$$L_0 = -\frac{1}{2e}\,\dot{x}_\mu \dot{x}^\mu - \frac{i}{2}\,\xi_\mu \dot{\xi}^\mu + \frac{i}{2e}\,\chi \dot{x}_\mu \xi^\mu,$$

$$L_m = -\frac{e}{2}\,m^2 + \frac{i}{2}\,\xi_5\,\dot{\xi}_5 + \frac{i}{2}\,m\chi\xi_5,$$

$$L_\theta = i\theta^{\dagger i} D^{ij}\theta^j + \frac{i}{2}\,eg\,Q^a F^a_{\mu\nu}\,\xi^\mu \xi^\nu.$$

Description in terms of Grassmann pseudovectors and pseudoscalars

In the absence of external Ψ-field Grassmann odd pseudovector ξ_μ appears in equation of motion only in bilinear (i.e. Grassmann-even) combination:

$$S^{\mu\nu} \equiv -i\,\xi^\mu \xi^\nu,$$

where $S_{\mu\nu}$ obeys the equation

$$\frac{dS^{\mu\nu}}{d\tau} = \frac{g}{m}\,Q^a (F^{a\mu}{}_\lambda S^{\lambda\nu} - F^{a\nu}{}_\lambda S^{\lambda\mu}).$$

$S_{\mu\nu}$ can be also represented by ψ_α spinors as:

$$S^{\mu\nu} = \frac{1}{2}\,\bar{\psi}\sigma^{\mu\nu}\psi.$$

Mapping construction

Let's consider a linear mapping:

$$\psi = \kappa\,\xi_\mu(\gamma^\mu\gamma_5\theta) + \alpha\xi_5(\gamma_5\theta),$$

and correspondingly for the conjugated function:

$$\bar\psi = -\kappa^*(\bar\theta\gamma_5\gamma^\mu)\xi_\mu - \alpha^*(\bar\theta\gamma_5)\xi_5.$$

Here, κ and α are unknown coefficient functions, $\theta = (\theta_\alpha)$ is an auxiliary Grassmann-odd Dirac spinor and the symbol $*$ is a complex conjugation sign.

$$\xi_\mu = \frac{1}{2}\left\{\beta(\bar\theta\gamma_\mu\gamma_5\psi) - \beta^*(\bar\psi\gamma_5\gamma_\mu\theta)\right\},$$
$$\xi_5 = \frac{1}{2}\left\{\tilde\beta(\bar\theta\gamma_5\psi) - \tilde\beta^*(\bar\psi\gamma_5\theta)\right\},$$

where β and $\tilde\beta$ are some new unknown coefficient functions.

Self-consistency equations

If we demand self-consistency of the mapping for $\xi_\mu = \xi_\mu(\bar{\psi}, \psi)$ it leads us to the following equations for the coefficient functions:

$$\begin{aligned} \frac{1}{2}(\beta\kappa + \beta^*\kappa^*)(\bar{\theta}\theta) &= 1, \\ \beta\kappa - \beta^*\kappa^* &= 0, \\ \beta\alpha + \beta^*\alpha^* &= 0, \end{aligned}$$

and correspondingly for $\xi_5 = \xi_5(\bar{\psi}, \psi)$

$$\begin{aligned} -\frac{1}{2}(\tilde{\beta}\alpha + \tilde{\beta}^*\alpha^*)(\bar{\theta}\theta) &= 1, \\ (\tilde{\beta}\kappa + \tilde{\beta}^*\kappa^*) &= 0. \end{aligned}$$

Kinetic term mapping

Let us now turn to a map of separate terms of the Lagrangian $L = L(\bar{\psi}, \psi)$ to the one $L = L(\xi_\mu, \xi_5)$. At first we consider a mapping of the kinetic term

$$\frac{1}{2i}\left(\frac{d\bar{\psi}}{d\tau}\,\psi - \bar{\psi}\,\frac{d\psi}{d\tau}\right).$$

It has a form:

$$
\begin{aligned}
&- i|\kappa|^2\,(\bar{\theta}\theta)\left(\xi_\mu\,\frac{d\xi^\mu}{d\tau}\right) - i|\alpha|^2(\bar{\theta}\theta)\left(\xi_5\,\frac{d\xi_5}{d\tau}\right)\\
&- \frac{1}{2}\left\{\frac{d\kappa^*}{d\tau}\,\kappa - \kappa^*\,\frac{d\kappa}{d\tau}\right\}(\theta\sigma^{\mu\nu}\theta)\,\xi_\mu\xi_\nu - \frac{1}{2}\,|\kappa|^2\left[\left(\frac{d\bar{\theta}}{d\tau}\,\sigma^{\mu\nu}\theta\right) - \left(\bar{\theta}\,\sigma^{\mu\nu}\frac{d\theta}{d\tau}\right)\right]\xi_\mu\,\xi_\nu\\
&+ \frac{1}{2i}\left\{\left(\kappa\frac{d\alpha^*}{d\tau} - \frac{d\kappa}{d\tau}\,\alpha^*\right) + \left(\kappa^*\frac{d\alpha}{d\tau} - \frac{d\kappa^*}{d\tau}\,\alpha\right)\right\}(\theta\gamma^\mu\theta)\,\xi_5\,\xi_\mu\\
&+ \frac{1}{2i}\,(\alpha^*\kappa + \alpha\kappa^*)(\bar{\theta}\gamma^\mu\theta)\left\{\frac{d\xi_5}{d\tau}\,\xi_\mu - \xi_5\,\frac{d\xi_\mu}{d\tau}\right\}\\
&+ \frac{1}{2i}\,(\alpha^*\kappa - \alpha\kappa^*)\left\{\left(\frac{d\bar{\theta}}{d\tau}\,\gamma^\mu\theta\right) - \left(\bar{\theta}\,\gamma^\mu\frac{d\theta}{d\tau}\right)\right\}\xi_5\,\xi_\mu.
\end{aligned}
$$

Coefficient functions

Here, the first two terms have exactly the same structure as the terms

$$-\frac{i}{2}\left(\xi_\mu \frac{d\xi^\mu}{d\tau}\right) \quad \text{and} \quad +\frac{i}{2}\left(\xi_5 \frac{d\xi_5}{d\tau}\right).$$

The requirement of literal coincidence of these two terms results in the algebraic equations

$$+|\kappa|^2(\bar{\theta}\theta) = \frac{1}{2}\,, -|\alpha|^2(\bar{\theta}\theta) = \frac{1}{2}\,.$$

But before comparison the second term, it is necessary preliminary to eliminate the χ-field by using of the equation of motion for ξ_5:

$$2\dot{\xi}_5 - m\chi = 0.$$

After such an elimination the kinetic term $\xi_5\dot{\xi}_5$ changes its sign to opposite one and comparing with appropriate terms results in the algebraic equation with the correct sign

$$+|\alpha|^2(\bar{\theta}\theta) = \frac{1}{2}\,.$$

Solving of the system on coefficient function

We seek a solution of the system of algebraic equations in the form $\kappa = ae^{i\varphi}$, a is unknown real function. The result is:

$$\kappa = (\pm)_\kappa \, \frac{1}{\sqrt{2}(\bar\theta\theta)^{1/2}} \, e^{i\varphi}, \quad \alpha = (\pm)_\alpha \, \frac{i(-1)^n}{\sqrt{2}(\bar\theta\theta)^{1/2}} \, e^{i\varphi},$$

$$\beta = (\pm)_\kappa \, \frac{\sqrt{2}}{(\bar\theta\theta)^{1/2}} \, e^{-i\varphi}, \qquad \tilde\beta = (\pm)_\alpha \, \frac{i\sqrt{2}(-1)^n}{(\bar\theta\theta)^{1/2}} \, e^{-i\varphi}.$$

here, $n = 0, \pm 1, \pm 2, \ldots$; φ is arbitrary phase, which is function of τ and $(\pm)_\kappa$, $(\pm)_\alpha$ denote arbitrariness in choose of signs independent for functions (κ, β) and $(\alpha, \tilde\beta)$.

Solution of the system

If we now substitute the obtained solutions in the remainder of the untapped coefficient functions, then we find expressions for them:

$$\frac{1}{2}\left\{\frac{d\kappa^*}{d\tau}\,\kappa-\kappa^*\,\frac{d\kappa}{d\tau}\right\}=\frac{1}{2i}\,\frac{1}{(\bar{\theta}\theta)}\,\frac{d\varphi}{d\tau}\,,$$

$$\frac{1}{2i}\left\{\left(\kappa\frac{d\alpha^*}{d\tau}-\frac{d\kappa}{d\tau}\,\alpha^*\right)+\left(\kappa^*\frac{d\alpha}{d\tau}-\frac{d\kappa^*}{d\tau}\,\alpha\right)\right\}=\pm\, i\,\frac{1}{(\bar{\theta}\theta)}\,\frac{d\varphi}{d\tau}\,,$$

$$\frac{1}{2i}\left(\alpha^*\kappa+\alpha\kappa^*\right)=0,\quad \frac{1}{2i}\left(\alpha^*\kappa-\alpha\kappa^*\right)=\mp\,\frac{1}{2(\bar{\theta}\theta)}\,.$$

Mapping problems

Requirement of coincidence for “force” term in the Lagrangians under mapping

$$-\frac{eg}{4}\,Q^a F^a_{\mu\nu}(\bar{\psi}\sigma^{\mu\nu}\psi) \sim \frac{ieg}{4}\,Q^a F^a_{\mu\nu}\,\xi^\mu\xi^\nu + \ldots$$

leads to the following condition:

$$-\,|\kappa|^2(\bar{\theta}\theta) = 1\,.$$

This equation contradicts with the one obtained before

$$\frac{dS^{\mu\nu}}{d\tau} = \frac{g}{m}\,Q^a(F^{a\mu}{}_{\lambda}S^{\lambda\nu} - F^{a\nu}{}_{\lambda}S^{\lambda\mu}).$$

Thus it is shown, that starting “naive” mapping is not full.

Full mapping

The analysis we have given shown that the mapping suggested is not one-to-one mapping. Thus we need to consider the most general mapping:

$$\psi = [\kappa \xi_\mu (\gamma^\mu \gamma_5 \theta) + \alpha \xi_5 (\gamma_5 \theta)] + \rho^* \zeta_{\mu\nu} (\sigma^{\mu\nu} \gamma_5 \theta) + \left[\hat{\kappa} \hat{\xi}_\mu (\gamma^\mu \theta) + \hat{\alpha} \hat{\xi}_5 \theta\right],$$

where

$$\begin{aligned}
\xi_\mu &= \frac{1}{2} \left\{ \beta(\bar{\theta} \gamma_\mu \gamma_5 \psi) - \beta^* (\bar{\psi} \gamma_5 \gamma_\mu \theta) \right\}, \\
\xi_5 &= \frac{1}{2} \left\{ \tilde{\beta}(\bar{\theta} \gamma_5 \psi) - \tilde{\beta}^* (\bar{\psi} \gamma_5 \theta) \right\}, \\
{}^*\zeta_{\mu\nu} &= \frac{1}{2} \left\{ s(\bar{\theta} \sigma_{\mu\nu} \gamma_5 \psi) - s^* (\bar{\psi} \gamma_5 \sigma_{\mu\nu} \theta) \right\}, \\
\hat{\xi}_\mu &= \frac{1}{2} \left\{ \hat{\beta}(\bar{\theta} \gamma_\mu \psi) - \hat{\beta}^* (\bar{\psi} \gamma_\mu \theta) \right\}, \\
\hat{\xi}_5 &= \frac{1}{2} \left\{ \hat{\tilde{\beta}}(\bar{\theta} \psi) - \hat{\tilde{\beta}}^* (\bar{\psi} \theta) \right\}.
\end{aligned}$$

Full mapping

If we wish to obtain a complete one-to-oneness of the mapping $(\psi, \bar{\psi}) \rightleftarrows (\xi_5,\, \xi_\mu,\, \ldots)$ we must use some additional suggestions. The simplest of them is to restrict the spinor ψ_α as well as θ_α to Majorana one, i.e. to require that they satisfy the condition

$$\psi = \psi^c, \qquad \theta = \theta^c. \tag{1}$$

The situation qualitatively changes in the presence of an external fermion Dirac field $\Psi^i_\alpha(x)$. It is clear that such a field inevitably violates the representations (1) for Majorana spinors. As known, a general Dirac spinor $\psi_{\rm D}$ can be always written in terms of two Majorana spinors

$$\psi_{\rm D} = \psi^{(1)}_{\rm M} + i\psi^{(2)}_{\rm M},$$

where

$$\psi^{(1)}_{\rm M} = \frac{1}{2}\,(\psi_{\rm D} + \psi^c_{\rm D}), \qquad \psi^{(2)}_{\rm M} = \frac{1}{2i}\,(\psi_{\rm D} - \psi^c_{\rm D}).$$

Such a decomposition can be performed both for the background fermion field $\Psi^i_\alpha(x)$ and for spinors ψ_α and θ_α. Then for each of Majorana spinors $\psi^{(i)}_{\rm M}$, $i = 1,\, 2$ we define own set of odd real currents $\xi^{(i)}_\mu,\, \xi^{(i)}_5,\, \xi^{(i)}_{\mu\nu},\, \ldots$.

Thank you for the attention!

Auxiliary spinor structure

General structure of axillary spinor has a form:

$$\theta_\alpha = \theta_\alpha(\theta^i, \theta^{\dagger i}, \vartheta^a, \Psi^i_\alpha, \bar{\Psi}^i_\alpha).$$

Performing the infinitesimal gauge transformations:

$$\begin{aligned} \theta^i &\rightarrow \theta^i + ig\Lambda^a (t^a)^{ij}\theta^j, \\ \vartheta^a &\rightarrow \vartheta^a - gf^{abc}\Lambda^b\vartheta^c, \\ \Psi^i_\alpha &\rightarrow \Psi^i_\alpha + ig\Lambda^a (t^a)^{ij}\Psi^j_\alpha, \end{aligned}$$

and demanding variation to be zero, one will get

$$\delta_\Lambda \theta_\alpha = ig\Lambda^a \left(\frac{\overrightarrow{\partial}\theta_\alpha}{\partial\theta^i}(t^a)^{ij}\theta^j - \theta^{\dagger j}(t^a)^{ji}\frac{\overleftarrow{\partial}\theta_\alpha}{\partial\theta^{\dagger i}} - \frac{\overrightarrow{\partial}\theta_\alpha}{\partial\vartheta^b}(T^b)^{ac}\vartheta^c + \right.$$
$$\left. + \frac{\overrightarrow{\partial}\theta_\alpha}{\partial\Psi^i_\alpha}(t^a)^{ij}\Psi^j_\alpha - \bar{\Psi}^j_\alpha (t^a)^{ji}\frac{\overleftarrow{\partial}\theta_\alpha}{\partial\bar{\Psi}^i_\alpha} \right) = 0.$$

Auxiliary spinor structure

If we represent the spinor as

$$\theta_\alpha = \mathcal{F}^{\dagger i}(\theta^\dagger, \theta, \vartheta)\Psi^i_\alpha,$$

then the condition of gauge invariance is reduced:

$$-\frac{\overrightarrow{\partial}\mathcal{F}^{\dagger i}}{\partial\theta^j}(t^a)^{jk}\theta^k + \theta^{\dagger k}(t^a)^{kj}\frac{\overleftarrow{\partial}\mathcal{F}^{\dagger i}}{\partial\theta^{\dagger j}} + \mathcal{F}^{\dagger j}(t^a)^{ji} + \frac{\overrightarrow{\partial}\mathcal{F}^{\dagger i}}{\partial\vartheta^b}(T^b)^{ac}\vartheta^c = 0.$$

It is easy to verify that the following simple combination of color charges:

$$\theta^{\dagger j}\vartheta^a(t^a)^{ji}$$

Auxiliary spinor structure

If now we consider $\mathcal{F}^{\dagger i}$ structure,

$$\mathcal{F}^{\dagger i} = \theta^{\dagger j} \vartheta^a (t^a f)^{ji},$$

we will get the following condition:

$$\frac{\overrightarrow{\partial} f^{ij}}{\partial \theta^k} (t^a)^{ks} \theta^s - \theta^{\dagger s} (t^a)^{sk} \frac{\overleftarrow{\partial} f^{ij}}{\partial \theta^{\dagger k}} - \frac{\overrightarrow{\partial} f^{ij}}{\partial \vartheta^b} (T^b)^{ac} \vartheta^c + (t^a)^{ik} f^{kj} - f^{ik} (t^a)^{kj} = 0.$$

This equation immediately lead to the wide range of solutions:

$$f^{ij} \sim \delta^{ij}, \quad \theta^i \theta^{\dagger j}, \quad (t^a \theta)^i (\theta^\dagger t^a)^j, \quad Q^a (t^a)^{ij}, \quad \mathcal{Q}^a (t^a)^{ij}, \ldots$$

Thus we find a broad class of possible expressions that can be taken as the required odd spinor: θ_α

$$\theta_\alpha \sim \vartheta^a (\theta^\dagger t^a \Psi_\alpha), \quad \vartheta^a Q^a (\theta^\dagger \Psi_\alpha), \quad \vartheta^a (\theta^\dagger t^a t^b \theta)(\theta^\dagger t^b \Psi_\alpha), \quad \vartheta^a Q^b (\theta^\dagger t^a t^b \Psi_\alpha) \ldots$$

Carrying out similar calculations for fermion field dependance, we get

$$\theta_\alpha \sim \theta^{\dagger j} \vartheta^a (t^a)^{ji} \left(\dot{x}^\mu D^{jk}_\mu (A) \Psi^k_\alpha (x) \right), \quad \theta^{\dagger j} \vartheta^a Q^b (t^a t^b)^{ji} \left(\dot{x}^\mu D^{jk}_\mu (A) \Psi^k_\alpha (x) \right),$$

Full mapping solution

Demanding one-to-one correspondence of the mapping, and paying special attention to contribution of $\zeta_{\mu\nu}$, one can finally get the system

$$|\kappa|^2(\bar{\theta}\theta) + 3\dot{x}^2|\rho|^2(\bar{\theta}\theta) - \frac{3i}{2}(\kappa^*\rho - \kappa\rho^*)\left[(\bar{\theta}\gamma^\lambda\theta)\dot{x}_\lambda\right] = \frac{1}{2},$$
$$|\kappa|^2(\bar{\theta}\theta) + 2\dot{x}^2|\rho|^2(\bar{\theta}\theta) - \frac{i}{2}(\kappa^*\rho - \kappa\rho^*)\left[(\bar{\theta}\gamma^\lambda\theta)\dot{x}_\lambda\right] = -1.$$

The system obtained is consistent at least at a formal level.