Standard Model

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Free real scalar field

Lagrangian

$$L = \frac{1}{2} (\partial_\mu \varphi) (\partial^\mu \varphi) - \frac{m^2}{2} \varphi^2$$

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Functional integral $m^2 \to m^2 - i0$

$$<T\{\varphi(x_1)\varphi(x_2)\}> = \frac{\int e^{iS(\varphi)} \varphi(x_1)\varphi(x_2) D\varphi}{\int e^{iS(\varphi)} D\varphi}$$

$$S(\varphi) = \int L d^4x \qquad D\varphi = \prod_x d\varphi(x)$$

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Generating functional

$$<T\{\varphi(x_1)\varphi(x_2)\}> = \left[\frac{1}{Z(J)}\frac{1}{i}\frac{\delta}{\delta J(x_1)}\frac{1}{i}\frac{\delta}{\delta J(x_2)}Z(J)\right]_{J=0}$$

$$Z(J) = \int e^{iS(\varphi,J)}D\varphi(x) \qquad S(\varphi,J) = \int (L + J\varphi)d^4x$$

Quadratic form

$$L = \frac{1}{2}\varphi \hat{M} \varphi \qquad \hat{M}(\partial) = -\partial^2 - m^2$$

Minimum of $S(\varphi, J)$

$$\hat{M}\varphi_0 + J = 0 \qquad \varphi_0 = -\hat{G}J \qquad \hat{G} = \hat{M}^{-1}$$

$$\varphi_0(x) = -\int G(x-y)J(y)d^4y \qquad \hat{M}G(x-y) = \delta(x-y)$$

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Momentum space

$$G(p) = M^{-1}(-ip) = \frac{1}{p^2 - m^2 + i0}$$

$$G(x) = \int G(p)e^{-ipx}\frac{d^4p}{(2\pi)^4}$$

Shift $\varphi = \varphi_0 + \varphi'$

$$S(\varphi, J) = \frac{1}{2} \int \left(-J\hat{G}J + \varphi' \hat{M} \varphi' \right) d^4x$$

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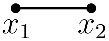
$$Z(J) = Z(0) \exp \left[-\frac{i}{2} \int J(x) G(x-y) J(y) \, d^4x \, d^4y \right]$$

Free propagator

$$<T\{\varphi(x_1)\varphi(x_2)\}> =$$



$$= iG(x_1 - x_2)$$

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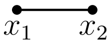
$$<T\{\varphi(x_1)\varphi(x_2)\}> = $$ 

$$= iG(x_1 - x_2)$$

Wick theorem

$$\begin{aligned}
&<T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\}> \\
&\quad = <T\{\varphi(x_1)\varphi(x_2)\}> <T\{\varphi(x_3)\varphi(x_4)\}> \\
&\quad + <T\{\varphi(x_1)\varphi(x_3)\}> <T\{\varphi(x_2)\varphi(x_4)\}> \\
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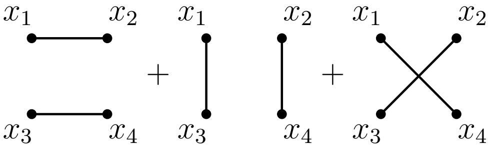
\end{aligned}$$

$=$ 

Interaction

$$L = L_0 + L_1 \qquad L_0 = \frac{1}{2}\left(\partial_\mu \varphi\right)\left(\partial^\mu \varphi\right) - \frac{m^2}{2}\varphi^2 \qquad L_1 = -\frac{\lambda}{4!}\varphi^4$$

Symmetry $\varphi \to -\varphi$

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Symmetry $\varphi \to -\varphi$

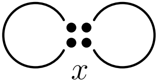
Perturbation theory

$$Z(J) = \int e^{iS_0(\varphi, J)}\left(1 + iS_1(\varphi) + \cdots\right) D\varphi$$

Vacuum $\to$ vacuum

$$<0|0> = <1>_0 - \frac{i}{4!}\lambda \int <\varphi^4(x)>_0 \, d^4x + \cdots$$

$$= 1 - \frac{i}{4!}\lambda$$



$$+ \cdots$$

Vacuum energy density $\to$ phase

Vacuum → vacuum

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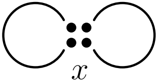
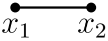
$$= 1 - \frac{i}{4!}\lambda$$ 

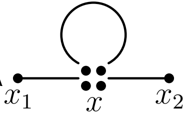
$$+ \cdots$$

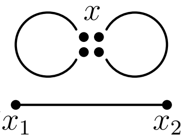
Vacuum energy density → phase

Propagator

$$\int e^{iS_0(\varphi)}(1 + iS_1(\varphi) + \cdots)\varphi(x_1)\varphi(x_2) D\varphi =$$



$$- \frac{i}{4!}\lambda$$ 

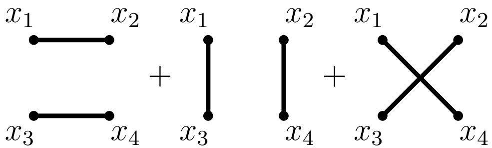
$$- \frac{i}{4!}\lambda$$ 

$$+ \cdots$$

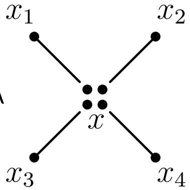
Diagrams with vacuum bubbles cancel

$$<T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\}>$$

$=$

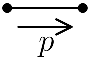


$-\frac{i}{4!}\lambda$

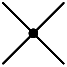


$+\cdots$

Feynman rules



$= iG_0(p) \qquad G_0(p) = \frac{1}{p^2 - m^2 + i0}$



$= -i\lambda$

Beware of symmetry factors!

Renormalization

Lagrangian

$$L = \frac{1}{2}\left(\partial_\mu \varphi_0\right)\left(\partial^\mu \varphi_0\right) - \frac{m_0^2}{2}\varphi_0^2 - \frac{\lambda_0}{4!}\varphi_0^4$$

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Renormalization

$$\varphi_0 = Z_\varphi^{1/2}\varphi \qquad m_0 = Z_m m \qquad \lambda_0 = Z_\lambda \lambda$$

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Renormalization

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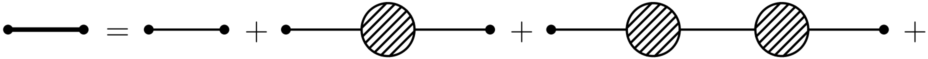
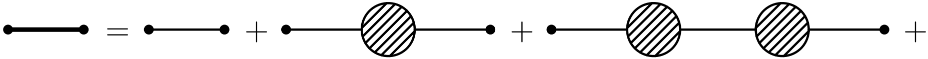
In $d = 4 - 2\varepsilon$ dimensions: $[L] = d$, $[\varphi] = 1 - \varepsilon$, $[\lambda] = 2\varepsilon$

Propagator



$$iG = iG_0 + iG_0(-i)\Sigma iG_0 + iG_0(-i)\Sigma iG_0(-i)\Sigma iG_0 + \cdots$$

Propagator

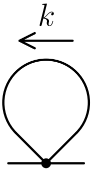


$$iG = iG_0 + iG_0(-i)\Sigma iG_0 + iG_0(-i)\Sigma iG_0(-i)\Sigma iG_0 + \cdots$$

$$G(p) = G_0(p) + G_0(p)\Sigma(p)G(p) \qquad G^{-1}(p) = G_0^{-1}(p) - \Sigma(p)$$

$$G(p) = \frac{1}{p^2 - m_0^2 - \Sigma(p) + i0}$$

1 loop

$-i\Sigma(p) =$ 

$$\Sigma(p) = -i\frac{\lambda_0}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{m_0^2 - k^2 - i0}$$

Symmetry factor $\frac{1}{2}$

1 loop

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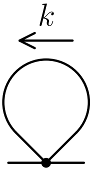

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$$\int \frac{d^dk}{(m^2 - k^2 - i0)^n} = i\pi^{d/2}m^{d-2n}V(n) \qquad V(n) = \frac{\Gamma\left(n - \frac{d}{2}\right)}{\Gamma(n)}$$

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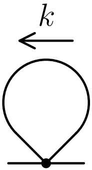
$$-i\Sigma(p) =$$ 

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$$\Sigma(p) = \frac{1}{2}\frac{\lambda_0 m_0^{d-2}}{(4\pi)^{d/2}}\Gamma\left(1 - \frac{d}{2}\right)$$

On-shell renormalization: mass

$$G(p) = \frac{1}{p^2 - m_0^2 - \Sigma(p^2) - i0}$$

has pole at $p^2 = m_{\rm os}^2$

$$m_{\rm os}^2 - m_0^2 - \Sigma(m_{\rm os}^2) = 0$$

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1 loop

$$m_0^2 = m_{\text{os}}^2 \left[1 + \frac{1}{d-2} \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) + \cdots \right]$$

$$Z_m^{\text{os}} = 1 + \frac{1}{2(d-2)} \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) + \cdots$$

On-shell renormalization: field

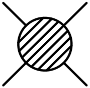
At $p^2 \to m_{\rm os}^2$

$$G_{\rm os}(p) \to G_0(p) = \frac{1}{p^2 - m_{\rm os}^2 + i0}$$

1 loop

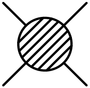
$$Z_\varphi^{\rm os} = 1 + \mathcal{O}(\lambda^2)$$

Vertex



$$= -i\lambda_0 \Gamma(p_i) \qquad \Gamma(p_i) = 1 + \Lambda(p_i)$$

Vertex



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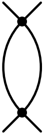
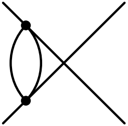
1 loop

$\Lambda(p_i) =$

$+$



$+$



$$= -\frac{1}{2}\frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}}\left[f\left(\frac{s}{m_0^2}\right) + f\left(\frac{t}{m_0^2}\right) + f\left(\frac{u}{m_0^2}\right)\right]$$

where

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{[m^2 - k^2 - i0]\,[m^2 - (k+p)^2 - i0]} = \frac{im^{-2\varepsilon}}{(4\pi)^{d/2}} f\left(\frac{p^2}{m^2}\right)$$

1 loop

$$f(0) = \Gamma(\varepsilon)$$

$f(x) - f(0)$ finite at $\varepsilon \to 0$

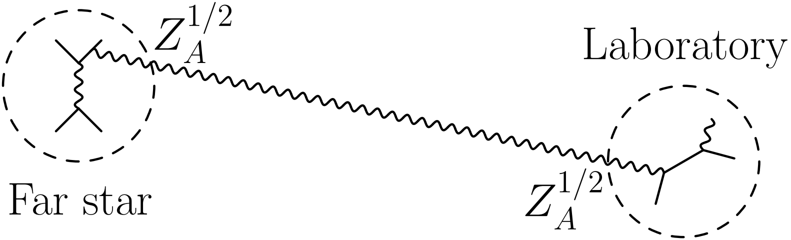
$$\Gamma = 1 - \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \left(\frac{3}{2} + \mathcal{O}(\varepsilon) \right)$$

$\overline{\text{MS}}$ renormalization

$$\frac{\lambda_0}{(4\pi)^{d/2}} = \mu^{2\varepsilon} \frac{\lambda(\mu)}{(4\pi)^2} Z_\lambda(\lambda(\mu)) e^{\gamma\varepsilon}$$

$$Z_\lambda(\lambda) = 1 + \frac{z_1}{\varepsilon} \frac{\lambda}{(4\pi)^2} + \left(\frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon} \right) \frac{\lambda^2}{(4\pi)^4} + \cdots$$

LSZ reduction formula



Free propagator $\Rightarrow$ spin wave functions
Full propagator (very) close to the mass shell

S-matrix element $=$ vertex $\times$ $(Z_i^{\mathrm{os}})^{1/2}$ for each i

Renormalization of the coupling constant

Vertex

$$\Gamma = Z_\Gamma \Gamma_r$$

Γ_r finite at $\varepsilon \to 0$

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Scattering amplitude

$$\lambda_0 \Gamma Z_\varphi^2 = Z_\lambda Z_\Gamma Z_\varphi^2 \lambda \Gamma_r$$

must be finite at $\varepsilon \to 0$ ($Z_\varphi^{\mathrm{os}} / Z_\varphi = \text{finite}$)

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$$Z_\lambda Z_\Gamma Z_\varphi^2 = \text{finite}$$

Renormalization of the coupling constant

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$$Z_\lambda Z_\Gamma Z_\varphi^2 = 1$$

$$Z_\lambda = \left(Z_\Gamma Z_\varphi^2\right)^{-1}$$

Renormalization group

$$Z_\lambda = 1 + z_1 \frac{\lambda}{(4\pi)^2 \varepsilon} + \cdots = 1 + \frac{3}{2} \frac{\lambda}{(4\pi)^2 \varepsilon} + \cdots$$

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λ_0 does not depend on μ

$$\frac{d \log \lambda}{d \log \mu} = -2\varepsilon + \frac{d \log Z_\lambda}{d \log \lambda} \frac{d \log \lambda}{d \log \mu} = -2\varepsilon + 2 z_1 \frac{\lambda}{(4\pi)^2} + \cdots$$

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Renormalization group equation at $\varepsilon = 0$

$$\frac{d}{d \log \mu} \frac{\lambda(\mu)}{(4\pi)^2} = \beta(\lambda(\mu))$$

$$\beta(\lambda) = \beta_0 \frac{\lambda^2}{(4\pi)^4} + \cdots$$

$$\beta_0 = 2 z_1 = 3$$

$\lambda(\mu)$ grows with μ