

# Standard Model

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# Free real scalar field

Lagrangian

$$L = \frac{1}{2} (\partial_\mu \varphi) (\partial^\mu \varphi) - \frac{m^2}{2} \varphi^2$$

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Functional integral  $m^2 \rightarrow m^2 - i0$

$$\langle T\{\varphi(x_1)\varphi(x_2)\} \rangle = \frac{\int e^{iS(\varphi)} \varphi(x_1)\varphi(x_2) D\varphi}{\int e^{iS(\varphi)} D\varphi}$$

$$S(\varphi) = \int L d^4x \quad D\varphi = \prod_x d\varphi(x)$$

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Generating functional

$$\langle T\{\varphi(x_1)\varphi(x_2)\} \rangle = \left[ \frac{1}{Z(J)} \frac{1}{i} \frac{\delta}{\delta J(x_1)} \frac{1}{i} \frac{\delta}{\delta J(x_2)} Z(J) \right]_{J=0}$$

$$Z(J) = \int e^{iS(\varphi, J)} D\varphi(x) \quad S(\varphi, J) = \int (L + J\varphi) d^4x$$

## Quadratic form

$$L = \frac{1}{2}\varphi\hat{M}\varphi \quad \hat{M}(\partial) = -\partial^2 - m^2$$

Minimum of  $S(\varphi, J)$

$$\hat{M}\varphi_0 + J = 0 \quad \varphi_0 = -\hat{G}J \quad \hat{G} = \hat{M}^{-1}$$

$$\varphi_0(x) = -\int G(x-y)J(y)d^4y \quad \hat{M}G(x-y) = \delta(x-y)$$

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## Momentum space

$$G(p) = M^{-1}(-ip) = \frac{1}{p^2 - m^2 + i0}$$

$$G(x) = \int G(p)e^{-ipx} \frac{d^4p}{(2\pi)^4}$$

Shift  $\varphi = \varphi_0 + \varphi'$

$$S(\varphi, J) = \frac{1}{2} \int \left( -J\hat{G}J + \varphi'\hat{M}\varphi' \right) d^4x$$

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$$Z(J) = Z(0) \exp \left[ -\frac{i}{2} \int J(x)G(x-y)J(y) d^4x d^4y \right]$$



# Free propagator

$$\langle T\{\varphi(x_1)\varphi(x_2)\}\rangle = \begin{array}{c} \bullet \text{---} \bullet \\ x_1 \quad x_2 \end{array} = iG(x_1 - x_2)$$

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## Wick theorem

$$\begin{aligned} \langle T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\}\rangle &= \langle T\{\varphi(x_1)\varphi(x_2)\}\rangle \langle T\{\varphi(x_3)\varphi(x_4)\}\rangle \\ &+ \langle T\{\varphi(x_1)\varphi(x_3)\}\rangle \langle T\{\varphi(x_2)\varphi(x_4)\}\rangle \\ &+ \langle T\{\varphi(x_1)\varphi(x_4)\}\rangle \langle T\{\varphi(x_2)\varphi(x_3)\}\rangle \end{aligned}$$

$$= \begin{array}{c} x_1 \quad x_2 \\ \bullet \text{---} \bullet \\ x_3 \quad x_4 \end{array} + \begin{array}{c} x_1 \\ \bullet \\ | \\ \bullet \\ x_3 \end{array} \begin{array}{c} x_2 \\ \bullet \\ | \\ \bullet \\ x_4 \end{array} + \begin{array}{c} x_2 \\ \bullet \\ | \\ \bullet \\ x_4 \end{array} \begin{array}{c} x_1 \quad x_2 \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ x_3 \quad x_4 \end{array}$$

# Interaction

$$L = L_0 + L_1 \quad L_0 = \frac{1}{2} (\partial_\mu \varphi) (\partial^\mu \varphi) - \frac{m^2}{2} \varphi^2 \quad L_1 = -\frac{\lambda}{4!} \varphi^4$$

Symmetry  $\varphi \rightarrow -\varphi$

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Symmetry  $\varphi \rightarrow -\varphi$

Perturbation theory

$$Z(J) = \int e^{iS_0(\varphi, J)} (1 + iS_1(\varphi) + \dots) D\varphi$$


Vacuum  $\rightarrow$  vacuum

$$\begin{aligned}\langle 0|0\rangle &= \langle 1\rangle_0 - \frac{i}{4!}\lambda \int \langle \varphi^4(x)\rangle_0 d^4x + \dots \\ &= 1 - \frac{i}{4!}\lambda \text{ (diagram)} + \dots\end{aligned}$$

The diagram shows two circles connected at a central point labeled  $x$ . At this point, there are four black dots representing the four legs of a  $\varphi^4$  vertex.

Vacuum energy density  $\rightarrow$  phase

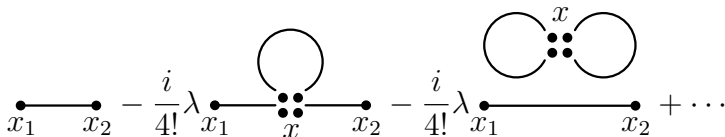
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Vacuum energy density  $\rightarrow$  phase

Propagator

$$\int e^{iS_0(\varphi)} (1 + iS_1(\varphi) + \dots) \varphi(x_1)\varphi(x_2) D\varphi =$$


$$- \frac{i}{4!}\lambda \text{ (diagram)} - \frac{i}{4!}\lambda \text{ (diagram)} + \dots$$

Diagrams with vacuum bubbles cancel

$$\langle T\{\varphi(x_1)\varphi(x_2)\varphi(x_3)\varphi(x_4)\}\rangle$$

$$= \begin{array}{c} x_1 \quad x_2 \\ \text{---} \\ x_3 \quad x_4 \end{array} + \begin{array}{c} x_1 \\ | \\ x_3 \end{array} \begin{array}{c} x_2 \\ | \\ x_4 \end{array} + \begin{array}{c} x_1 \quad x_2 \\ \diagdown \quad \diagup \\ x_3 \quad x_4 \end{array}$$

$$- \frac{i}{4!} \lambda \begin{array}{c} x_1 \quad x_2 \\ \diagdown \quad \diagup \\ \bullet \bullet \\ x \\ \bullet \bullet \\ \diagup \quad \diagdown \\ x_3 \quad x_4 \end{array} + \dots$$

# Feynman rules

$$\begin{array}{l} \bullet \text{---} \bullet \quad = \quad iG_0(p) \\ \quad \quad \quad \underbrace{\quad \quad}_{\rightarrow p} \\ \\ \times \quad = \quad -i\lambda \end{array} \quad G_0(p) = \frac{1}{p^2 - m^2 + i0}$$

Beware of symmetry factors!



# Renormalization

Lagrangian

$$L = \frac{1}{2} (\partial_\mu \varphi_0) (\partial^\mu \varphi_0) - \frac{m_0^2}{2} \varphi_0^2 - \frac{\lambda_0}{4!} \varphi_0^4$$

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## Lagrangian

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$$\varphi_0 = Z_\varphi^{1/2} \varphi \quad m_0 = Z_m m \quad \lambda_0 = Z_\lambda \lambda$$

# Renormalization

## Lagrangian

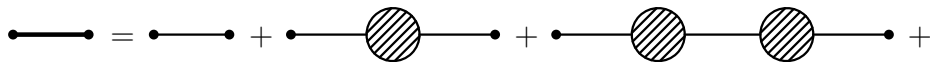
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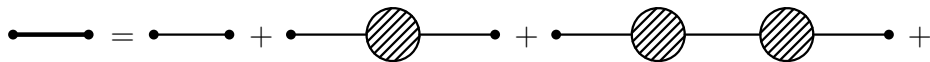
In  $d = 4 - 2\varepsilon$  dimensions:  $[L] = d$ ,  $[\varphi] = 1 - \varepsilon$ ,  $[\lambda] = 2\varepsilon$

# Propagator



$$iG = iG_0 + iG_0(-i)\Sigma iG_0 + iG_0(-i)\Sigma iG_0(-i)\Sigma iG_0 + \dots$$

# Propagator

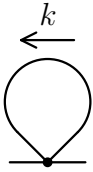


$$iG = iG_0 + iG_0(-i)\Sigma iG_0 + iG_0(-i)\Sigma iG_0(-i)\Sigma iG_0 + \dots$$

$$G(p) = G_0(p) + G_0(p)\Sigma(p)G(p) \quad G^{-1}(p) = G_0^{-1}(p) - \Sigma(p)$$

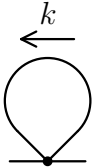
$$G(p) = \frac{1}{p^2 - m_0^2 - \Sigma(p) + i0}$$

# 1 loop


$$-i\Sigma(p) =$$
$$\Sigma(p) = -i\frac{\lambda_0}{2} \int \frac{d^d k}{(2\pi)^d} \frac{1}{m_0^2 - k^2 - i0}$$

Symmetry factor  $\frac{1}{2}$

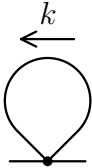
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Symmetry factor  $\frac{1}{2}$

$$\int \frac{d^d k}{(m^2 - k^2 - i0)^n} = i\pi^{d/2} m^{d-2n} V(n) \quad V(n) = \frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)}$$

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$$\Sigma(p) = \frac{1}{2} \frac{\lambda_0 m_0^{d-2}}{(4\pi)^{d/2}} \Gamma(1 - \frac{d}{2})$$



## On-shell renormalization: mass

$$G(p) = \frac{1}{p^2 - m_0^2 - \Sigma(p^2) - i0}$$

has pole at  $p^2 = m_{\text{os}}^2$

$$m_{\text{os}}^2 - m_0^2 - \Sigma(m_{\text{os}}^2) = 0$$

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1 loop

$$m_0^2 = m_{\text{os}}^2 \left[ 1 + \frac{1}{d-2} \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) + \dots \right]$$

$$Z_m^{\text{os}} = 1 + \frac{1}{2(d-2)} \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) + \dots$$

# On-shell renormalization: field

At  $p^2 \rightarrow m_{\text{os}}^2$

$$G_{\text{os}}(p) \rightarrow G_0(p) = \frac{1}{p^2 - m_{\text{os}}^2 + i0}$$

1 loop

$$Z_{\varphi}^{\text{os}} = 1 + \mathcal{O}(\lambda^2)$$

# Vertex

$$\text{Diagram: a circle with diagonal hatching and two lines crossing it} = -i\lambda_0\Gamma(p_i) \quad \Gamma(p_i) = 1 + \Lambda(p_i)$$

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$$\text{Diagram: a circle with diagonal hatching and four external lines} = -i\lambda_0\Gamma(p_i) \quad \Gamma(p_i) = 1 + \Lambda(p_i)$$

1 loop

$$\begin{aligned} \Lambda(p_i) &= \text{Diagram: bubble with two external lines} + \text{Diagram: bubble with two external lines and a vertical line} + \text{Diagram: bubble with two external lines and a diagonal line} \\ &= -\frac{1}{2} \frac{\lambda_0 m_0^{-2\epsilon}}{(4\pi)^{d/2}} \left[ f\left(\frac{s}{m_0^2}\right) + f\left(\frac{t}{m_0^2}\right) + f\left(\frac{u}{m_0^2}\right) \right] \end{aligned}$$

where

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{[m^2 - k^2 - i0][m^2 - (k+p)^2 - i0]} = \frac{im^{-2\epsilon}}{(4\pi)^{d/2}} f\left(\frac{p^2}{m^2}\right)$$

# 1 loop

$$f(0) = \Gamma(\varepsilon)$$

$f(x) - f(0)$  finite at  $\varepsilon \rightarrow 0$

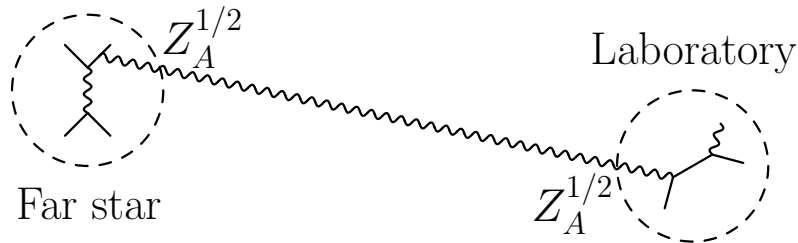
$$\Gamma = 1 - \frac{\lambda_0 m_0^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) \left( \frac{3}{2} + \mathcal{O}(\varepsilon) \right)$$

# $\overline{\text{MS}}$ renormalization

$$\frac{\lambda_0}{(4\pi)^{d/2}} = \mu^{2\varepsilon} \frac{\lambda(\mu)}{(4\pi)^2} Z_\lambda(\lambda(\mu)) e^{\gamma\varepsilon}$$

$$Z_\lambda(\lambda) = 1 + \frac{z_1}{\varepsilon} \frac{\lambda}{(4\pi)^2} + \left( \frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon} \right) \frac{\lambda^2}{(4\pi)^4} + \dots$$

# LSZ reduction formula



Free propagator  $\Rightarrow$  spin wave functions

Full propagator (very) close to the mass shell

$S$ -matrix element = vertex  $\times (Z_i^{\text{os}})^{1/2}$  for each  $i$



# Renormalization of the coupling constant

Vertex

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Scattering amplitude

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$$Z_\lambda Z_\Gamma Z_\varphi^2 = 1$$

$$Z_\lambda = (Z_\Gamma Z_\varphi^2)^{-1}$$

# Renormalization group

$$Z_\lambda = 1 + z_1 \frac{\lambda}{(4\pi)^2 \varepsilon} + \dots = 1 + \frac{3}{2} \frac{\lambda}{(4\pi)^2 \varepsilon} + \dots$$

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$\lambda_0$  does not depend on  $\mu$

$$\frac{d \log \lambda}{d \log \mu} = -2\varepsilon + \frac{d \log Z_\lambda}{d \log \lambda} \frac{d \log \lambda}{d \log \mu} = -2\varepsilon + 2z_1 \frac{\lambda}{(4\pi)^2} + \dots$$

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Renormalization group equation at  $\varepsilon = 0$

$$\frac{d}{d \log \mu} \frac{\lambda(\mu)}{(4\pi)^2} = \beta(\lambda(\mu))$$

$$\beta(\lambda) = \beta_0 \frac{\lambda^2}{(4\pi)^4} + \dots$$

$$\beta_0 = 2z_1 = 3$$

$\lambda(\mu)$  grows with  $\mu$