When are neutrino oscillations observable?

Another source of decoherence: wave packet separation due to the difference of group velocities Δv of different mass eigenstates.

If coherence is lost: Flavour transition can still occur, but in a non-oscillatory way. E.g. for $\pi \to \mu\nu_i$ decay with a subsequent detection of ν_i with the emission of e:

$$P \;\propto\; \sum_i P_{\rm prod}(\mu\,\nu_i)P_{\rm det}(e\,\nu_i) \;\propto\; \sum_i |U_{\mu i}|^2|U_{ei}|^2$$

– the same result as for averaged oscillations.

How are the oscillations destroyed? Suppose by measuring momenta and energies of particles at neutrino production (or detection) we can determine its energy E and momentum p with uncertainties σ_E and σ_p. From $E_i^2 \;=\; p_i^2 + m_i^2$:

$$\sigma_{m^2} \;=\; \left[(2E\sigma_E)^2 \;+\; (2p\sigma_p)^2\right]^{1/2}$$

When are neutrino oscillations observable?

If $\sigma_{m^2} < \Delta m^2 = |m_i^2 - m_k^2|$ – one can tell which mass eigenstate is emitted.

$\sigma_{m^2} < \Delta m^2$ implies $2p\sigma_p < \Delta m^2$, or $\sigma_p < \Delta m^2/2p \simeq l_{\rm osc}^{-1}$.

<u>But:</u> To measure p with the accuracy σ_p one needs to measure the momenta of particles at production with (at least) the same accuracy $\Rightarrow$ uncertainty of their coordinates (and the coordinate of ν production point) will be

$$\sigma_{\rm x,\,prod} \gtrsim \sigma_p^{-1} > l_{\rm osc}$$

$\Rightarrow$ Oscillations washed out. Similarly for neutrino detection.

Natural <u>necessary</u> condition for coherence (observability of oscillations):

$$L_{\rm source} \ll l_{\rm osc}\,, \qquad L_{\rm det} \ll l_{\rm osc}$$

No averaging of oscillations in the source and detector

Satisfied with very large margins in most cases of practical interest

Wave packet separation

Wave packets representing different mass eigenstate components have different group velocities v_{gi} $\Rightarrow$ after time $t_{\rm coh}$ (coherence time) they separate $\Rightarrow$ Neutrinos stop oscillating! (Only averaged effect observable).

Coherence time and length:

$$\Delta v \cdot t_{\rm coh} \simeq \sigma_x \,; \qquad\qquad l_{\rm coh} \simeq v t_{\rm coh}$$

$$\Delta v = \frac{p_i}{E_i} - \frac{p_k}{E_k} \simeq \frac{\Delta m^2}{2E^2}$$

$$\boxed{l_{\rm coh} \simeq \frac{v}{\Delta v}\sigma_x = \frac{2E^2}{\Delta m^2}\, v\sigma_x}$$

The standard formula for $P_{\rm osc}$ is obtained when the decoherence effects are negligible.

A manifestation of neutrino coherence

Even non-observation of neutrino oscillations at distances $L \ll l_{\rm osc}$ is a consequence of and an evidence for coherence of neutrino emission and detection! Two-flavour example (e.g. for ν_e emission and detection):

$$A_{\rm prod/det}(\nu_1) \sim \cos\theta\,, \qquad A_{\rm prod/det}(\nu_2) \sim \sin\theta \qquad \Rightarrow$$

$$A(\nu_e \to \nu_e) = \sum_{i=1,2} A_{\rm prod}(\nu_i) A_{\rm det}(\nu_i) \sim \cos^2\theta + e^{-i\Delta\phi}\sin^2\theta$$

Phase difference $\Delta\phi$ vanishes at short L $\Rightarrow$

$$P(\nu_e \to \nu_e) = (\cos^2\theta + \sin^2\theta)^2 = 1$$

If ν_1 and ν_2 were emitted and absorbed incoherently) $\Rightarrow$ one would have to sum probabilities rather than amplitudes:

$$P(\nu_e \to \nu_e) \sim \sum_{i=1,2} |A_{\rm prod}(\nu_i) A_{\rm det}(\nu_i)|^2 \sim \cos^4\theta + \sin^4\theta \; < \; 1$$

Neutrino oscillations: *Coherence at macroscopic distances –*
$L > 10{,}000$ km in atmospheric neutrino experiments!

Oscillations and QM uncertainty relations

Neutrino oscillations – a QM interference phenomenon, owe their existence to QM uncertainty relations

Neutrino energy and momentum are characterized by uncertainties σ_E and σ_p related to the spatial localization and time scale of the production and detection processes. These uncertainties

- allow the emitted/absorbed neutrino state to be a coherent superposition of different mass eigenstates
- determine the size of the neutrino wave packets $\Rightarrow$ govern decoherence due to wave packet separation

σ_E – the effective energy uncertainty, dominated by the smaller one between the energy uncertainties at production and detection. Similarly for σ_p.

The paradox of σ_E and σ_p

QM uncertainty relations: σ_p is related to the spatial localization of the production (detection) process, while σ_E to its time scale $\Rightarrow$ independent quantities.

On the other hand: Neutrinos propagating macroscopic distances are on the mass shell. For on-shell mass eigenstates $E^2 = p^2 + m_i^2$ means

$$E\sigma_E = p\sigma_p$$

How can this be understood?

The solution: At production, neutrinos are *not* on the mass shell. They go on shell only after they propagate $x \sim$ (a few)$\times$ De Broglie wavelengths. After that their energy and momentum get related by $E^2 = p^2 + m_i^2 \quad \Rightarrow$ the larger uncertainty shrinks towards the smaller one to satisfy $E\sigma_E = p\sigma_p$.

On-shell relation between E and p allows to determine the less certain of the two through the more certain one, reducing the error of the former.

What determines the length of ν w. packets?

The length of ν w. packets: $\sigma_x \sim 1/\sigma_p$. For propagating on-shell neutrinos:

$$\sigma_p \simeq \min\{\sigma_p^{\rm prod},\, (E/p)\sigma_E^{\rm prod}\} = \min\{\sigma_p^{\rm prod},\, (1/v_g)\sigma_E^{\rm prod}\}$$

Which uncertainty is smaller at production, $\sigma_p^{\rm prod}$ or $\sigma_E^{\rm prod}$?

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- If $T_S < \tau$ (τ – lifetime of the parent unstable particle) $\Rightarrow$ $\sigma_E \simeq T_S^{-1}$ (collisional broadening). Mom. uncertainty: $\sigma_p \simeq L_S^{-1}$.

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- If $T_S > \tau$ (quasi-free parent particle) $\Rightarrow$ $\sigma_E \simeq \tau^{-1} = \Gamma$.

  $\sigma_p \simeq [(p/E)\tau]^{-1} \simeq [(p/E)\sigma_E]^{-1}$, i.e. $\sigma_E \simeq (p/E)\sigma_p < \sigma_p$.

The length of ν w. packets – contd.

In both cases $\boxed{\sigma_E^{\rm prod} < \sigma_p^{\rm prod}}$ $\Leftarrow$ also when $\nu' s$ are produced in collisions.

$$\Longrightarrow \qquad \sigma_{p\ \rm eff} \simeq \frac{\sigma_E}{v_g}\,, \qquad\qquad \boxed{\sigma_x \simeq \frac{v_g}{\sigma_E}}$$

In the stationary limit $(\sigma_E \to 0)$ one has $\sigma_{p\ \rm eff} \to 0$ even though σ_p is finite!
Therefore $\sigma_x \to \infty$ and so the coherence length $l_{\rm coh} \to \infty$
– a well known result.

Lorentz invariance issues

1. "Paradox" of neutrino w. packet length

For neutrino production in decays of unstable particles at rest (e.g. $\pi \to \mu\nu_\mu$):

$$\sigma_E \simeq \tau^{-1} = \Gamma_\pi\,, \qquad \sigma_x \simeq \frac{v_g}{\sigma_E} \simeq \frac{v_g}{\Gamma_\pi} \ (= v_g \tau)$$

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For decay in flight: $\Gamma'_\pi = (m_\pi/E_\pi)\Gamma_\pi$. One might expect

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On the other hand, if the decaying pion is boosted in the direction of the neutrino momentum, the neutrino w. packet should be Lorentz-contracted !

The solution: pion decay takes finite time. During the decay time the pion moves over distance $l = u\tau'$ ("chases" the neutrino if $u > 0$).

$$\sigma'_x \simeq v'_g/\Gamma' - l = v'_g\tau' - u\tau' = (v'_g - u)\gamma_u\tau = \frac{v_g\tau}{\gamma_u(1 + v_g u)}\,,$$

[the relativ. law of addition of velocities: $v'_g = (v_g + u)/(1 + v_g u)$].

Lorentz invariance issues – contd.

That is

$$\sigma'_x = \frac{\sigma_x}{\gamma_u(1+v_g u)}$$

For relativistic neutrinos $v_g \approx v'_g \approx 1 \;\; \Rightarrow$

$$\sigma'_x = \sigma_x \sqrt{\frac{1-u}{1+u}}$$

$\Rightarrow$ when the pion is boosted in the direction of neutrino emission $(u>0)$ the neutrino wave packet gets contracted; when it is boosted in the opposite direction $(u<0)$ – the wave packet gets dilated.

Lorentz invariance issues – contd.

The oscillation probability must be Lorentz invariant! But: L. invariance is not obvious in QM w. packet approach which (unlike QFT) is not manifestly Lorentz covariant.

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How can we see Lorentz invariance of the standard formula for the oscillation probability? P_{ab} depends on L/p (contains factors $\exp[-i\frac{\Delta m_{ik}^2}{2p}L]$). Is L/p Lorentz invariant?

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$$L' = \gamma_u(L + ut)\,, \qquad t' = \gamma_u(t + uL)\,,$$

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The stand. osc. formula results when (i) production and detection and (ii) propagation are coherent; for neutrinos from conventional sources (i) implies $\sigma_x \ll l_{\rm osc}$. $\Rightarrow$ one can consider neutrinos pointlike and set $L = v_g t$. $\Rightarrow$ $L' = \gamma_u L(1 + u/v_g)$.

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$\Rightarrow$

$$\boxed{L'/p' = L/p}$$

Lorentz invariance issues – contd.

A more general argument (applies also to Mössbauer neutrinos which are not pointlike): Consider the phase difference

$$\diamondsuit \qquad \Delta\phi = -\frac{1}{v_g}(L - v_g\, t)\Delta E + \frac{\Delta m^2}{2p}\, L$$

– a Lorentz invariant quantity, though the two terms are in not in general separately Lorentz invariant.

But: If the 1st term is negligible in all Lorentz frames, the second term is Lorentz invariant by itself $\Rightarrow$ L/p is Lorentz invariant.

The 1st term can be neglected when the production/detection coherence conditions are satisfied. In particular, it vanishes in the limit of pointlike neutrinos $L = v_g t$. N.B.:

$$L' - v_g' t' = \gamma_u \left[(L + ut) - \frac{v_g + u}{1 + v_g u}(t + uL)\right] = \frac{L - v_g t}{\gamma_u (1 + v_g u)}\,,$$

i.e. the condition $L = v_g t$ is Lorentz invariant. MB neutrinos: $\Delta E \simeq 0$.

Lorentz invariance issues – contd.

The oscillation probability must be Lorentz invariant even when the coherence conditions are not satisfied !
Lorentz invariance is enforced by the normalization condition.

$$P_{ab}(L) \;=\; \sum_{i,k} U_{ai}U_{bi}^{*}U_{ak}^{*}U_{bk}\, I_{ik}(L)\,, \qquad \text{where}$$

$$I_{ik}(L) \;\equiv\; \int dT\, \mathcal{A}_i(L,T)\mathcal{A}_k^{*}(L,T)e^{-i\Delta\phi_{ik}}$$

From the norm. cond. $\int dT\, |\mathcal{A}_i(L,T)|^2 \;=\; 1 \qquad \Rightarrow$

$$|\mathcal{A}_i|^2 dt \;=\; inv. \quad\Rightarrow\quad |\mathcal{A}_i||\mathcal{A}_k|dt \;=\; inv. \quad\Rightarrow\quad \mathcal{A}_i\mathcal{A}_k^{*}dT \;=\; inv.$$

The phase difference $\Delta\phi_{ik} = \Delta E_{ik}T - \Delta p_{ik}L$ is also Lorentz invariant $\Rightarrow$ so is $I_{ik}(L)$, and consequently $P_{ab}(L)$.

Shortcomings of the QM w. packet approach

- Neutrino wave packet postulated rather than derived, widths estimated
- Production and detection processes are not considered
- Inadequate normalization procedure. Normalization “by hand” is unavoidable.

Advantage: simplicity

Calc. from 1st principles – QFT approach

Production - propagation - detection treated as a single inseparable process. External particles are described by wave packets, neutrinos – by propagators

One-particle states of external particles:

$$|A\rangle = \int [dp]\, f_A(\vec{p}, \vec{P})\, |A, \vec{p}\rangle\,, \qquad\qquad [dp] \equiv \frac{d^3p}{(2\pi)^3\sqrt{2E_A(\vec{p})}}$$

$|A, \vec{p}\rangle$ – one-particle momentum eigenstate corresponding to momentum $\vec{p}$ and energy $E_A(\vec{p})$ (free particles: $E_A(\vec{p}) = \sqrt{\vec{p}^2 + m_A^2}$). The normalization condition for the plane wave states $|A, \vec{p}\rangle$:

$$\langle A, \vec{p}'|A, \vec{p}\rangle = 2E_A(\vec{p})\,(2\pi)^3\delta^{(3)}(\vec{p} - \vec{p}')\,.$$

$f_A(\vec{p}, \vec{P})$ – momentum distribution function with the mean momentum $\vec{P}$.
Normalization condition: $\langle A|A\rangle = 1 \quad \Rightarrow \quad \int d^3p\, |f_A(\vec{p})|^2/(2\pi)^3 = 1$.

QFT approach – contd.

Coordinate-space wave packet with maximum at $\vec{x} = \vec{x}_0$ at the time $t - t_0$:

$$\Psi_A(x) = \int [dp]\, f_A(\vec{p}) e^{-iE_A(\vec{p})(t-t_0)+i\vec{p}(\vec{x}-\vec{x}_0)}$$

Consistent with the usual QFT definition of the wave function:

$$\Psi_A(x) = \langle 0|\hat{\Psi}_A(x)|A\rangle \,.$$

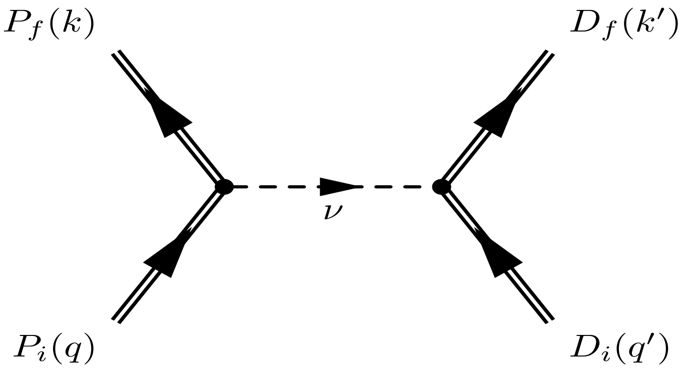
Transition amplitude:

$$\mathcal{A}_{\alpha\beta} \;=\; \sum_j U^*_{\alpha j} U_{\beta j} \mathcal{A}_j \,.$$

Use the Feynman rules in the configuration space. In lowest (2nd) order in weak interaction:

$$\mathcal{A}_j = \int d^4x_1 \int d^4x_2\, A_j^P(x_1) S_{Fj}(x_1 - x_2) A_j^D(x_2) \,.$$

How is it obtained?



$$|P_i\rangle = \int [dq]\, f_{Pi}(\vec{q}, \vec{Q})\, |P_i, \vec{q}\rangle\,, \qquad |P_f\rangle = \int [dk]\, f_{Pf}(\vec{k}, \vec{K})\, |P_f, \vec{k}\rangle\,,$$

$$|D_i\rangle = \int [dq']\, f_{Di}(\vec{q}', \vec{Q}')\, |D_i, \vec{q}'\rangle\,, \qquad |D_f\rangle = \int [dk']\, f_{Df}(\vec{k}', \vec{K}')\, |D_f, \vec{k}'\rangle\,.$$

The transition amplitude:

$$i\mathcal{A}_{\alpha\beta} = \langle P_f\, D_f | \hat{T} \exp\Big[-i \int d^4x\, \mathcal{H}_I(x) \Big] - \mathbb{1} | P_i\, D_i \rangle\,,$$

QFT approach – contd.

In the second order in weak interaction:

$$i\mathcal{A}_{\alpha\beta} = \sum_j U_{\alpha j}^* U_{\beta j} \int [dq]\, f_{Pi}(\vec{q},\vec{Q}) \int [dk]\, f_{Pf}^*(\vec{k},\vec{K}) \times \int [dq']\, f_{Di}(\vec{q}\,',\vec{Q}') \int [dk']\, f_{Df}^*(\vec{k}',\vec{K}')\, i\mathcal{A}_j^{p.w.}(q,k;q',k')\,.$$

Plane-wave amplitude:

$$i\mathcal{A}_j^{p.w.}(q,k;q',k') = \int d^4x_1 \int d^4x_2\, \tilde{M}_D(q',k')\, e^{-i(q'-k')(x_2-x_D)} \times i\int \frac{d^4p}{(2\pi)^4} \frac{\not{p}+m_j}{p^2-m_j^2+i\epsilon}\, e^{-ip(x_2-x_1)}\, \tilde{M}_P(q,k)\, e^{-i(q-k)(x_1-x_P)}$$

$\tilde{M}_{jP}$, $\tilde{M}_{jD}$ – production and detection amplitudes with neutrino spinors excluded. Full amplitudes:

$$M_{jP}(q,k) \equiv \frac{\bar{u}_{jL}(p)}{\sqrt{2p_0}} \tilde{M}_P(q,k)\,, \qquad M_{jD}(q',k') \equiv \tilde{M}_D(q',k') \frac{u_{jL}(p)}{\sqrt{2p_0}}$$

QFT approach – contd.

Neutrino prod. and det. regions: the overlap regions of the wave packets of participating external particles. 4-coordinates of the "central points" of these regions (points of the maximal overlap of external w. packets): x_P and x_D. It will be convenient to go to shifted 4-coordinates:

$$x_1' = x_1 - x_P\,, \qquad x_2' = x_2 - x_D\,.$$

Also define

$$T = t_D - t_P\,, \qquad \vec{L} = \vec{x}_D - \vec{x}_P\,.$$

A useful formula:

$$\not{p} + m_j = \sum_\sigma u_{j\sigma}(p)\bar{u}_{j\sigma}(p)\,.$$

For neutrinos only one chirality contributes $(\sigma = L$ for ν and $\sigma = R$ for $\bar{\nu})$ because of the chiral nature of weak interactions $\Rightarrow$ the sum over σ can be dropped; $u_{j\sigma}(p)$ and $\bar{u}_{j\sigma}(p)$ can then be merged with $\tilde{M}_{P,D}$ to produce M_{jP} and M_{jD}.

QFT approach – contd.

$$iA_{\alpha\beta} = i\sum_j U^*_{\alpha j}U_{\beta j}\int\frac{d^4p}{(2\pi)^4}\,\Phi_{jP}(p^0,\vec{p})\Phi_{jD}(p^0,\vec{p})\,\frac{2p_0\,e^{-ip^0T+i\vec{p}\vec{L}}}{p^2-m_j^2+i\epsilon}\,.$$

$$\Phi_{jP}(p^0,\vec{p}) = \int d^4x_1' e^{ipx_1'}\int[dq]\int[dk]\,f_{Pi}(\vec{q},\vec{Q})\,f^*_{Pf}(\vec{k},\vec{K})\,e^{-i(q-k)x_1'}M_{jP}(q,k)$$

$$\Phi_{jD}(p^0,\vec{p}) = \int d^4x_2' e^{-ipx_2'}\int[dq']\int[dk']\,f_{Di}(\vec{q}',\vec{Q}')f^*_{Df}(\vec{k}',\vec{K}')\,e^{-i(q'-k')x_2'}M_{jD}(q',k')$$

For $L \gg 1/p$ – fast oscillating factor in $iA_{\alpha\beta}$ $\Rightarrow$ main contribution to integral over p^0 from the pole at $p^0 = E_j(\vec{p}) - i\epsilon$ (on-shell neutrinos).

$$\Downarrow$$

$$iA_{\alpha\beta} = \Theta(T)\sum_j U^*_{\alpha j}U_{\beta j}\int\frac{d^3p}{(2\pi)^3}\,\Phi_{jP}(E_j(\vec{p}),\vec{p})\Phi_{jD}(E_j(\vec{p}),\vec{p})\,e^{-iE_j(\vec{p})T+i\vec{p}\vec{L}}$$

In the QM w.packet approach we had:

Transition amplitude

$$\mathcal{A}_{\alpha\beta}(T,\vec{L}) \;=\; \langle \nu_\beta^{\rm fl}|\nu_\alpha^{\rm fl}(T,\vec{L})\rangle \;=\; \sum_j U_{\alpha j}^* U_{\beta j}\,\mathcal{A}_j(T,\vec{L})$$

$$\boxed{\mathcal{A}_j(T,\vec{L}) \;=\; \int \frac{d^3p}{(2\pi)^3}\, f_j^S(\vec{p})\, f_j^{D*}(\vec{p})\, e^{-iE_j(p)T+i\vec{p}\vec{L}}}$$

The QM and QFT expressions have exactly the same form !

QFT approach – contd.

Comparing with $\mathcal{A}_{ab}(T,\vec{L})$ obtained in the QM w. packet approach: the two amplitudes coincide if

$$f_{jP}(\vec{p}) = \Phi_{jP}(E_j(\vec{p}),\vec{p})\,, \qquad f_{jD}(\vec{p}) = \Phi^{*}_{jD}(E_j(\vec{p}),\vec{p})\,,$$

QFT approach – contd.

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Easy to understand: $\Phi_{jP}(E_j(p),\vec{p})$ is the probability amplitude of ν production process in which ν_j is emitted with momentum $\vec{p}$

$\Rightarrow$ Φ_{jP} is momentum distribution function of the produced neutrino, i.e. the momentum-state wave packet $f_{jP}(\vec{p})$. Similarly for neutrino detection.

N.B.: $f_{jP}(\vec{p})$ and $f_{jD}(\vec{p})$ are not “canonically” normalized.

Alternative approaches:

QFT approach – contd.

Comparing with $\mathcal{A}_{ab}(T,\vec{L})$ obtained in the QM w. packet approach: the two amplitudes coincide if

$$f_{jP}(\vec{p}) = \Phi_{jP}(E_j(\vec{p}),\vec{p})\,, \qquad f_{jD}(\vec{p}) = \Phi^*_{jD}(E_j(\vec{p}),\vec{p})\,,$$

Easy to understand: $\Phi_{jP}(E_j(p),\vec{p})$ is the probability amplitude of ν production process in which ν_j is emitted with momentum $\vec{p}$

$\Rightarrow$ Φ_{jP} is momentum distribution function of the produced neutrino, i.e. the momentum-state wave packet $f_{jP}(\vec{p})$. Similarly for neutrino detection.

N.B.: $f_{jP}(\vec{p})$ and $f_{jD}(\vec{p})$ are not “canonically” normalized.

Alternative approaches:

- $|P_f\nu_j\rangle \;=\; (S-\mathbb{1})|P_i\rangle\,, \qquad |\nu_j\rangle \;=\; \langle P_f|P_f\nu_j\rangle$

QFT approach – contd.

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All three approaches give the same results.

General properties of ν w. packets in QFT

$$f_{jP}(\vec{p}) \simeq M_{jP}(Q,K)\int d^4x\, e^{iE_j(\vec{p})t - i\vec{p}\vec{x}}\int[dq]\int[dk] f_{Pi}(\vec{q},\vec{Q}) f^*_{Pf}(\vec{k},\vec{K}) e^{-i(q-k)x}$$

Integral over $\vec{x}$ gives $\sim \delta^{(3)}(\vec{q}-\vec{k}-\vec{p})$. Since $f_{Pi}(\vec{q},\vec{Q})$, $f_{Pf}(\vec{k},\vec{K})$ are sharply peaked at $\vec{Q}$ and $\vec{K}$ $\Rightarrow$ $f_{jP}(\vec{p})$ is sharply peaked at

$\vec{P} \equiv \vec{Q} - \vec{K}$. Width of the peak: $\sigma_{pP} \simeq \max\{\sigma_{P_i}, \sigma_{P_f}\}$

For external particles described by plane waves:

$$f_{jP}(\vec{p}) = \frac{M_{jP}(Q,K)}{\sqrt{2E_{Pi}V\cdot 2E_{Pf}V}}\,\delta^{(4)}(Q-K-p)$$

In general: $f_{jP}(\vec{p}) \Rightarrow M_{jP}(Q,K) \times$ ("smeared δ-functions") representing approx. conservation of mean energies and mean momenta.

Matching QM & QFT expressions for ν w. p.

Example – Gaussian wave packets for external particles. QFT gives

$$f_{jP}(\vec{p}) \;\propto\; [M_{jP}(Q,K)]/(\sigma_{eP}\sigma_{pP}^3)\;\exp\big[-g_P(E_j(\vec{p}),\,\vec{p})\big],$$

$$g_P(E_j(\vec{p}),\,\vec{p}) = \frac{(\vec{p}-\vec{P})^2}{4\sigma_{pP}^2} + \frac{[E_j(\vec{p})-E_P-\vec{v}_P(\vec{p}-\vec{P})]^2}{4\sigma_{eP}^2}\,.$$

Here $\quad \vec{P} \equiv \vec{Q}-\vec{K}\,, \quad E_P \equiv E_{Pi}(\vec{Q}) - E_{Pf}(\vec{K})\,,$

$$\sigma_{pP}^2 = \sigma_{pPi}^2 + \sigma_{pPf}^2\,, \qquad \sigma_{xP}\sigma_{pP} = \frac{1}{2}\,,$$

$$\vec{v}_P \equiv \sigma_{xP}^2\left(\frac{\vec{v}_{Pi}}{\sigma_{xPi}^2} + \frac{\vec{v}_{Pf}}{\sigma_{xPf}^2}\right), \qquad \Sigma_P \equiv \sigma_{xP}^2\left(\frac{\vec{v}_{Pi}^{\,2}}{\sigma_{xPi}^2} + \frac{\vec{v}_{Pf}^{\,2}}{\sigma_{xPf}^2}\right),$$

$$\sigma_{eP}^2 = \sigma_{pP}^2(\Sigma_P - \vec{v}_P^{\,2}) \equiv \sigma_{pP}^2\,\lambda_P\,, \qquad 0 \le \lambda_P \le 1\,.$$

For 2 ext. particles at production: $\sigma_{eP} = |\vec{v}_{Pi} - \vec{v}_{Pf}|/2\sqrt{\sigma_{xPi}^2 + \sigma_{xPf}^2} \sim$ inverse overlap time

Matching QM & QFT expressions for ν w. p.

Compare with Gaussian wave packet in QM approach:

$$f_{jP}(\vec{p},\vec{P}) = \left(\frac{2\pi}{\sigma_{pP}^2}\right)^{3/4} \exp\left[-\frac{(\vec{p}-\vec{P})^2}{4\sigma_{pP}^2}\right]$$

To match the QM and QFT expression: expand $E_j(\vec{p})$ around $\vec{p}=\vec{P}$ and subst. into $g_P(E_j(\vec{p}),\vec{p})$:

$\diamondsuit$ $$\boxed{g_P(E_j(\vec{p}),\vec{p}) = (p-P)^k\,\alpha^{kl}\,(p-P)^l - \beta^k(p-P)^k + \gamma_j}$$

$$\alpha^{kl} = \frac{1}{4\sigma_{eP}^2}\left[\lambda_P\,\delta^{kl} + (v_j - v_P)^k\,(v_j - v_P)^l + \frac{E_j - E_P}{E_j}(\delta^{kl} - v_j^k v_j^l)\right],$$

$$\beta^k = -\frac{1}{2\sigma_{eP}^2}(E_j - E_P)(v_j - v_P)^k\,, \qquad \gamma_j = \frac{(E_j - E_P)^2}{4\sigma_{eP}^2}\,.$$

Try to represent $g_P(E_j(\vec{p}),\vec{p})$ in the form

$\diamondsuit$ $$\boxed{g_P(E_j(\vec{p}),\vec{p}) = (p-P_{\rm eff})^k\,\alpha^{kl}\,(p-P_{\rm eff})^l + \tilde{\gamma}_j}\,, \qquad \vec{P}_{\rm eff} \equiv \vec{P} + \vec{\delta}$$

Matching QM & QFT expressions for ν w. p.

$$\delta^k = -\frac{(E_j - E_P)(v_j - v_P)^k}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}\,, \qquad \tilde{\gamma}_j = \frac{(E_j - E_P)^2}{4\sigma_{eP}^2}\,\frac{\lambda_P}{\lambda_P + (\vec{v}_j - \vec{v}_P)^2}\,.$$

Diagonalization of α^{kl} gives ($OZ||(\vec{v}_j - \vec{v}_P)$):

$$(\sigma^x_{pP\,\mathrm{eff}})^2 = (\sigma^y_{pP\,\mathrm{eff}})^2 = \sigma_{pP}^2\,, \qquad \frac{1}{(\sigma^z_{pP\,\mathrm{eff}})^2} = \frac{1}{\sigma_{pP}^2} + \frac{(\vec{v}_j - \vec{v}_P)^2}{\sigma_{eP}^2}\,,$$

$\Rightarrow$ QM neutrino wave packets can match those obtained QFT if

Matching QM & QFT expressions for ν w. p.

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$\Rightarrow$ QM neutrino wave packets can match those obtained QFT if

- Momentum uncertainties of the neutrino mass eigenstates are replaced (anisotropic) effective ones: $-(\vec{p} - \vec{P})^2/(4\sigma^2_{pP}) \;\to$

$-[(p^x - P^x_{\mathrm{eff}})^2/4(\sigma^x_{pP})^2 + (p^y - P^y_{\mathrm{eff}})^2/4(\sigma^y_{pP})^2 + (p^z - P^z_{\mathrm{eff}})^2/4(\sigma^z_{pP})^2]$.

Matching QM & QFT expressions for ν w. p.

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Matching QM & QFT expressions for ν w. p.

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  $-[(p^x - P_{\mathrm{eff}}^x)^2/4(\sigma_{pP}^x)^2 + (p^y - P_{\mathrm{eff}}^y)^2/4(\sigma_{pP}^y)^2 + (p^z - P_{\mathrm{eff}}^z)^2/4(\sigma_{pP}^z)^2]$.
- The mean momentum $\vec{P}$ is shifted according to $\vec{P} \rightarrow \vec{P}_{\mathrm{eff}} = \vec{P} + \vec{\delta}$.
- The wave packet of each neutrino mass eigenstate gets an extra factor $N_j = \exp[-\tilde{\gamma}_j]$.

Matching QM & QFT expressions for ν w. p.

If $$|E_i - E_j| \ll \sigma_{eP} \qquad \Rightarrow$$

factors N_j are the same for all ν mass eigenstates, can be included in common normalization factor. In the opposite case – coherence of different neutrino mass eigenstates is lost.

$\sigma_{eP} \le \sigma_{pP} \quad \Rightarrow$ except for $\vec{v}_j \approx \vec{v}_P$ momentum uncertainty along $(\vec{v}_j - \vec{v}_P)$ is dominated by σ_{eP}.

In the stationary neutrino source limit $(\sigma_{eP}\,, \vec{v}_P \to 0)$, effective longitudinal mom. uncertainty $\sigma^z_{pP\,\mathrm{eff}} = 0$ even though the true mom. uncertainty $\sigma_{pP} \neq 0$.

$$\Downarrow$$

Coherence length $l_{\mathrm{coh}} \;\to\; \infty$

Oscillation probability in QFT

What is calculated in QFT is the probability of the overall production-propagation-detection process. How to extract from it the oscillation probability $P_{\alpha\beta}(L)$?

1. Recall the operational definition of $P_{\alpha\beta}(L)$. Detection rate for ν_β:

$$\Gamma_\beta^{\rm det} = \int dE\, j_\beta(E)\sigma_\beta(E)\,,$$

If a source at a distance L from the detector emits ν_α with the energy spectrum $d\Gamma_\alpha^{\rm prod}(E)/dE$:

$$j_\beta(E) = \frac{1}{4\pi L^2}\frac{d\Gamma_\alpha^{\rm prod}(E)}{dE}P_{\alpha\beta}(L,E)\,,$$

$\Rightarrow$ substitute into $\Gamma_\beta^{\rm det}$:

Oscillation probability in QFT

$$\Gamma_{\alpha\beta}^{\text{tot}} \equiv \int dE\, \frac{d\Gamma_{\alpha\beta}^{\text{tot}}(E)}{dE} = \frac{1}{4\pi L^2}\int dE\, \frac{d\Gamma_{\alpha}^{\text{prod}}(E)}{dE}\, P_{\alpha\beta}(L,E)\, \sigma_\beta(E)$$

$$P_{\alpha\beta}(L,E) = \frac{d\Gamma_{\alpha\beta}^{\text{tot}}(E)/dE}{\frac{1}{4\pi L^2}\left[d\Gamma_{\alpha}^{\text{prod}}(E)/dE\right]\sigma_\beta(E)}\,.$$

An important ingredient: the assumption that the overall rate factorizes into the production rate, propagation (oscillation) probability and detection cross section.

If this does not hold, oscillation probability is undefined $\Rightarrow$

Need to deal instead with the overall rate of neutrino production, propagation and detection.

Oscillation probability in QFT

Try to cast $P^{\rm tot}_{\alpha\beta}$ in the same form (check if the factorization condition holds !)

$$i\mathcal{A}_{\alpha\beta} = i\sum_j U^*_{\alpha j}U_{\beta j}\int \frac{d^4p}{(2\pi)^4}\,\Phi_{jP}(p^0,\vec{p})\Phi_{jD}(p^0,\vec{p})\,\frac{2p_0\,e^{-ip^0T+i\vec{p}\vec{L}}}{p^2-m_j^2+i\epsilon}$$

Integrate first over $\vec{p}$, then over $p^0\equiv E$. Make use of Grimus-Stockinger theorem: for a large L ($L\gg p/\sigma_p^2$), $A>0$ and a sufficiently smooth $\psi(\vec{p})$,

$$\int d^3p\frac{\psi(\vec{p})\,e^{i\vec{p}\vec{L}}}{A-\vec{p}^2+i\epsilon} = -\frac{2\pi^2}{L}\psi(\sqrt{A}\tfrac{\vec{L}}{L})e^{i\sqrt{A}L}+\mathcal{O}(L^{-\frac{3}{2}}) \quad \Rightarrow$$

$$i\mathcal{A}_{\alpha\beta}(T,\vec{L}) = \frac{-i}{8\pi^2L}\sum_j U^*_{\alpha j}U_{\beta j}\int dE\,\Phi_P(E,p_j\vec{l})\Phi_D(E,p_j\vec{l})\;2E\,e^{-iE\,T+ip_jL}$$

where

$$p_j\equiv\sqrt{E^2-m_j^2}\,,\qquad \vec{l}\equiv\frac{\vec{L}}{L}\,,$$

Oscillation probability in QFT

Introduce

$$\tilde{P}^{\rm tot}_{\alpha\beta}(\vec{L}) = \int dT\, P_{\alpha\beta}(T,\vec{L}) = \frac{1}{8\pi^2}\,\frac{1}{4\pi L^2}\sum_{j,k} U^*_{\alpha j}U_{\beta j}U_{\alpha k}U^*_{\beta k}$$

$$\times \int dE\, \Phi_P(E,p_j\vec{l})\Phi_D(E,p_j\vec{l})\,\Phi^*_P(E,p_k\vec{l})\Phi^*_D(E,p_k\vec{l})\ (2E)^2\, e^{i(p_j-p_k)L}$$

Neutrino production probability:

$$P^{\rm prod}_{\alpha} = \sum_j |U_{\alpha j}|^2 \int \frac{d^3p_j}{(2\pi)^3}\,\big|\Phi_P(E,p_j)\big|^2 = \sum_j |U_{\alpha j}|^2 \frac{1}{8\pi^2}\int dE\,\big|\Phi_P(E,p_j)\big|^2 4Ep_j$$

Detection probability:

$$P^{\rm det}_{\beta}(E) = \sum_k |U_{\beta k}|^2 |\Phi_D(E,p_k)|^2 \frac{1}{V}\,,$$

Oscillation probability in QFT

Let the number of particles P_i entering the production region during time interval T_0 be N_P and number of D_i entering the detection region be N_D. Probability of neutrino emission during the finite interval of time t:

$$\mathcal{P}_\alpha^{\rm prod}(t) = N_P \int_0^t \frac{dt_P}{T_0}\, P_\alpha^{\rm prod} = N_P\, P_\alpha^{\rm prod}\frac{t}{T_0}\,, \quad \text{rate:} \quad \Gamma_\alpha^{\rm prod} = N_P\, P_\alpha^{\rm prod}\,\frac{1}{T_0}$$

Detection cross section:

$$\sigma_\beta(E) = \frac{N_D}{T_0}\sum_k |U_{\beta k}|^2 |\Phi_{kD}(E)|^2 \frac{E}{p_k}$$

Probability of the overall production-propagation-detection process:

$$\mathcal{P}_{\alpha\beta}^{\rm tot}(t,L) = \frac{N_P N_D}{T_0^2}\int_0^t dt_D \int_0^t dt_P\, P_{\alpha\beta}^{\rm tot}(T,L) \quad \Rightarrow$$

Oscillation probability in QFT

New integration variables $\tilde{T} \equiv (t_P + t_D)/2$ and $T = t_D - t_P \quad \Rightarrow$

$$\mathcal{P}^{\rm tot}_{\alpha\beta}(t, L) = \frac{N_P N_D}{T_0^2}\left[\int_0^t dT\, P^{\rm tot}_{\alpha\beta}(T, L)(t - T) + \int_{-t}^0 dT\, P^{\rm tot}_{\alpha\beta}(T, L)(t + T)\right]$$

$$= \frac{N_P N_D}{T_0^2}\left[t\int_{-t}^t dT\, P^{\rm tot}_{\alpha\beta}(T, L) - \int_0^t dT\, T\, P^{\rm tot}_{\alpha\beta}(T, L) + \int_{-t}^0 dT\, T\, P^{\rm tot}_{\alpha\beta}(T, L)\right]$$

$$\equiv \frac{N_P N_D}{T_0^2}\left[t I_1(t) - I_2(t) + I_3(t)\right].$$

For large t (much larger than the time scales of the neutrino production and detection processes) $I_1 = \tilde{P}^{\rm tot}_{\alpha\beta}(L)$ whereas $I_2 = I_3 = 0 \quad \Rightarrow$

$$\mathcal{P}^{\rm tot}_{\alpha\beta}(t, L) = \frac{N_P N_D}{T_0^2}\, t\, \tilde{P}^{\rm tot}_{\alpha\beta}(L)\,, \qquad \Gamma^{\rm tot}_{\alpha\beta}(L) = \frac{N_P N_D}{T_0^2}\, \tilde{P}^{\rm tot}_{\alpha\beta}$$

Oscillation probability in QFT

$$``P_{\alpha\beta}(L,E)" = \frac{\sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^* \Phi_P(E,p_j)\Phi_D(E,p_j)\Phi_P^*(E,p_k)\Phi_D^*(E,p_k) e^{i(p_j-p_k)L}}{\sum_j |U_{\alpha j}|^2 \, |\Phi_P(E,p_j)|^2 \, p_j \, \sum_k |U_{\beta k}|^2 \, |\Phi_D(E,p_k)|^2 p_k^{-1}}$$

For $|p_j - p_k| \ll p_j, p_k$ (ultra-relativistic or quasi-degenerate in mass ν's) and in addition $|p_j - p_k| \ll \sigma_p$: In expressions for $\Gamma_\alpha^{\rm prod}$ and σ_β can replace

$$p_j \to p\,, \qquad \Phi_P(E,p_j) \to \Phi_P(E,p) \qquad (p \;-\; \text{average momentum})$$

$\Rightarrow$ in the denominator of $``P_{\alpha\beta}(L,E)"$:

$$\sum_j |U_{\alpha j}|^2 \, |\Phi_P(E,p_j)|^2 \, p_j \to |\Phi_P(E,p)|^2 \, p \sum_j |U_{\alpha j}|^2 = |\Phi_P(E,p)|^2 \, p\,,$$

$$\sum_k |U_{\beta j}|^2 \, |\Phi_D(E,p_k)|^2 \, p_k^{-1} \to |\Phi_D(E,p)|^2 \, p^{-1} \sum_k |U_{\beta k}|^2 = |\Phi_D(E,p)|^2 \, p^{-1}\,,$$

The same can be done in the numerator of $``P_{\alpha\beta}(L,E)"$

Oscillation probability in QFT

For $|p_j - p_k| \ll \sigma_p$ $\Gamma_\alpha^{\rm prod}$ and σ_β do not depend on the elements of the mixing matrix $\Rightarrow$ factorization holds. $P_{\alpha\beta}(E,L)$ can be defined as a sensible quantity:

$$P_{\alpha\beta}(L,E) = \frac{|\Phi_P(E,p)|^2\,|\Phi_D(E,p)|^2 \sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^* e^{i(p_j - p_k)L}}{|\Phi_P(E,p)|^2\,|\Phi_D(E,p)|^2}$$

Automatically satisfies unitarity, i.e. is properly normalized.

For $|p_j - p_k| \gg \sigma_p$ $(\Leftrightarrow \Delta m_{jk}^2/(2p) \gg \sigma_p)$ – interf. terms in "$P_{\alpha\beta}(L,E)$" are strongly suppressed. In the opposite case

$$\frac{\Delta m_{jk}^2}{2p} \ll \sigma_p\,,$$

(production & detection coherence cond. satisfied) – $\Phi_P(E,p_{j,k})$, $\Phi_D(E,p_{j,k})$ can be pulled out of the sums $\Rightarrow$ stand. osc. probability:

$$P_{\alpha\beta}(L,E) = \sum_{j,k} U_{\alpha j}^* U_{\beta j} U_{\alpha k} U_{\beta k}^*\, e^{-i\frac{\Delta m_{jk}^2}{2p}L}$$

Oscillation probability in QFT

The conditions for the existence of well-defined universal oscillation probabilities is that neutrinos are either ultra-relativistic or nearly degenerate in mass and that neutrino production and detection are coherent.

The QFT-based consideration clarifies the QM wave packet normalization prescription. QM and QFT approaches can be matched if the QM quantities f_{jP} and f_{jD} are identified with the QFT functions $\Phi_{jP}(E_j, \vec{p})$ and $\Phi^*_{jD}(E_j, \vec{p})$, respectively. But: the latter bear information not only on the properties of the emitted and absorbed neutrinos, but also on the production and detection processes. The QM normalization procedure is equivalent, in the limit $|p_j - p_k| \ll p_j, p_k, \sigma_p$, to the division of the overall rate of the process by the production rate and detection cross section, as in QFT approach.

Do charged leptons oscillate?

What do we mean by charged leptons?

The usual $e^{\pm}$, $\mu^{\pm}$ and $\tau^{\pm}$ are mass eigenstates $\Rightarrow$ do not oscillate.

[Also: unlike neutrinos, they participate also in EM interactions (and are normally detected via these interactions) which are flavour-blind.]

Assume we create a muon at $t_0 = 0$ and $\vec{x}_0 = 0$. Neglecting muon decay, we have

$$|\Psi(0)\rangle = |\mu\rangle\,; \quad |\Psi(\vec{x},t)\rangle = e^{-ip_\mu x}|\mu\rangle \quad \Rightarrow \quad P_{\mu\mu} = |\langle\mu|\Psi(\vec{x},t)\rangle|^2 = 1$$

Assume now we manage to create a coherent superposition of μ and e:

$$|\Psi(0)\rangle = \cos\theta|\mu\rangle + \sin\theta|e\rangle$$

The weights of μ and e in the initial state: $\cos^2\theta$ and $\sin^2\theta$.

Do charged leptons oscillate?

Evolved state:

$$|\Psi(\vec{x}, t)\rangle \;=\; e^{-ip_\mu x}\cos\theta|\mu\rangle \;+\; e^{-ip_e x}\sin\theta|e\rangle$$

The probabilities of finding μ and e:

$$P_\mu \;=\; |\langle\mu|\Psi(\vec{x}\,, t)\rangle|^2 \;=\; |e^{-ip_\mu x}\cos\theta|^2 \;=\; \cos^2\theta$$

$$P_e \;=\; |\langle e|\Psi(\vec{x}\,, t)\rangle|^2 \;=\; |e^{-ip_e x}\sin\theta|^2 \;=\; \sin^2\theta$$

– are the same! $\Rightarrow$ There are no oscillations between mass eigenstates, no matter if the initial state is pure or (coherently) mixed

$\Downarrow$

There are no oscillations between e, μ and τ !

[NB: The same for neutrinos – initially produced ν_e can with some probability oscillate into ν_μ or ν_τ, but the weights of ν_1, ν_2 and ν_3 that were in the initial state will remain the same!]

Is that the full answer?

Can we imagine a situation when one creates a coherent superposition of e, μ and τ and then also detects their coherent superposition (the same or different) rather than individual mass-eigenstate charged leptons?

Charged - current weak interactions look completely symmetric w.r.t. neutrinos and charged leptons!

$$\mathcal{L}_{\rm CC} = -\frac{g}{\sqrt{2}}\left(\bar{e}_{aL}\gamma^{\mu}U_{ai}\nu_{iL}\right)W_{\mu}^{-} + h.c., \qquad U = V_L^{\dagger}V_{\nu}$$

Why do we say that charged leptons are emitted and detected in mass eigenstates and neutrinos in flavour states (superpositions of mass eigenstates) and not vice versa? Or not both as some superpositions of mass eigenstates? E.g.

$|e_1\rangle = U_{1e}|e\rangle + U_{1\mu}|\mu\rangle + U_{1\tau}|\tau\rangle$ is emitted or detected together with ν_1,
$|e_2\rangle = U_{2e}|e\rangle + U_{2\mu}|\mu\rangle + U_{2\tau}|\tau\rangle$ is emitted or detected together with ν_2,
$|e_3\rangle = U_{3e}|e\rangle + U_{3\mu}|\mu\rangle + U_{3\tau}|\tau\rangle$ is emitted or detected together with ν_3.

Why do neutrinos oscillate?

Because they are emitted (and absorbed) alongside charged leptons of definite mass $e^\pm$, $\mu^\pm$ or $\tau^\pm$. (This "measures" the flavour of neutrinos). How do we know that charged leptons are in mass eigenstates?

(1) Beta decay: only electrons are emitted together with neutrinos. Emission of $\mu^\pm$ and $\tau^\pm$ is forbidden by energy conservation.

(2) Decays $\pi^\pm \to \mu^\pm\nu$, $\pi^\pm \to e^\pm\nu$ (or $K^\pm \to \mu^\pm\nu$, $K^\pm \to e^\pm\nu$). Here emission of both muons and electrons is allowed.

Assume a coherent superposition of e and μ is produced in pion decay (nearly) at rest. The energy uncertainty of the charged lepton:

$$\sigma_E \simeq \Gamma_\pi = 2.5 \cdot 10^{-8}\ \mathrm{eV}$$

Uncertainty in the mass determination ($\sqrt{(2E\sigma_E)^2 + (2p\sigma_p)^2]} \simeq 2\sqrt{2}E\sigma_E$):

$$\sigma_{m^2} \sim 2\sqrt{2}E\sigma_E \simeq 2\sqrt{2} \cdot (90\ \mathrm{MeV}) \cdot (2.5 \cdot 10^{-8}\ \mathrm{eV}) \simeq 6.4\ \mathrm{eV}^2$$

Do charged leptons oscillate?

This has to be compared with $m_\mu^2 - m_e^2 \simeq (106\ \mathrm{MeV})^2 \quad \Rightarrow$

Different mass-eigenstate charged leptons are emitted incoherently!

This provides a "measurement" of the flavour of the emitted neutrino

For pion decay in flight: assume pion's energy is E_0. The energies of the produced charged leptons are rescaled as $E \to E\,(E_0/m_\pi)$, but the pion decay width (and so σ_E) is rescaled as $\Gamma_\pi \to \Gamma_\pi(m_\pi/E_0) \quad \Rightarrow$ $[(2E\sigma_E)^2 + (2p\sigma_p)^2]^{1/2}$ remains the same (σ_{m^2} a Lorentz invariant quantity).

$$\Downarrow$$

- ♢ Charged leptons produced in $\pi^\pm \to l^\pm \nu$ and $K^\pm \to l^\pm \nu$ decays are always emitted as mass eigenstates and not as coherent superpositions of different mass eigenstates because of their very large Δm^2.

- ♢ Therefore even oscillations between e_1, e_1 and e_3 (or any other superpositions of e, μ and τ) are not possible.

Do charged leptons oscillate?

The masses and decay widths of $\pi^\pm$, $K^\pm$ are rather small $\Rightarrow$ σ_{m^2} small.

How about decays of $W^\pm$? For $W^\pm \to l^\pm \nu$ decays at rest:

$$\Gamma^0_{W\to l_a \nu} \simeq \frac{G_F m_W^3}{6\sqrt{2}\pi} \simeq 230\ \text{MeV}$$

$$\Rightarrow \qquad \sigma_{m^2} \sim 2\sqrt{2}\,E\sigma_E \simeq 2\sqrt{2}\cdot 40\ \text{GeV}\cdot 230\ \text{MeV} \simeq (5\ \text{GeV})^2\,.$$

Thus

$$\sigma_{m^2} \gg m_\mu^2 - m_e^2\,, \qquad \sigma_{m^2} > m_\tau^2 - m_\mu^2 \simeq (1.77\ \text{GeV})^2\,,$$

$\Rightarrow$ all three charged leptons are produced *coherently* in $W^\pm$ decays.

Can one then observe oscillations between their different coh. superpositions?

Coherence length $l_{\text{coh}} \simeq \sigma_x/\Delta v_g$:

$$(l_{\text{coh}})_{\max} \simeq [\Gamma^0_{W\to l_a\nu}(\Delta v_g)_{\min}]^{-1} \simeq \frac{3\sqrt{2}\pi}{G_F m_W (m_\mu^2 - m_e^2)} \simeq 2.5\times 10^{-8}\ \text{cm}\,.$$

$\Rightarrow$ $l^\pm$ loose their coherence almost immediately after their production

Do charged leptons oscillate?

What about $W^\pm \to l^\pm \nu$ decays in flight? Let γ be the Lorentz factor of $W^\pm$. $(\Delta v_g)_{\rm min} \simeq \Delta m^2_{\mu e}/2E^2 \equiv (m_\mu^2 - m_e^2)/2E^2$ and the partial decay width of $W^\pm$ scale with γ as

$$(\Delta v_g)_{\rm min} \to \gamma^{-2} (\Delta v_g)_{\rm min}\,, \qquad \Gamma^0_{W\to l_a \nu} \to \gamma^{-1} \Gamma^0_{W\to l_a \nu}\,.$$

Therefore the maximum coherence length $(l_{\rm coh})_{\rm max} \simeq \sigma_x/(\Delta v_g)_{\rm min} \simeq 1/[\Gamma^0_{W\to l_a\nu}(\Delta v_g)_{\rm min}]$ scales as

$$(l_{\rm coh})_{\rm max} \to \gamma^3 (l_{\rm coh})_{\rm max}\,.$$

In order for $(l_{\rm coh})_{\rm max}$ to be larger than e.g. 1 m, one would need $\gamma \gtrsim 1600$, or $E_W \gtrsim 130$ TeV – far above presently feasible energies.

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N.B.: Even if coherence was satisfied for charged leptons, to fix the composition of the mixed $l^\pm$ state in terms of e, μ and τ one would have to detect the accompanying neutrino as a state different from $\nu_{\rm fl}$ – e.g. as a mass eigenstate. Not possible within the standard model!

Extensions of the standard model?

Consider the SM amended by three heavy RH neutrinos N_i (seesaw model) plus an extra Higgs doublet. In this model N_i can decay into a charged lepton and charged Higgs boson:

$$N_i \to e_i^- + \Phi^+ \,.$$

Decays are caused by the Yukawa coupling Lagrangian

$$\mathcal{L}_Y = Y_{ai} \bar{L}_a N_{Ri} \Phi \,+\, h.c.\,,$$

In the basis where the mass matrices of N_i and $l^{\pm}$ have been diagonalized, the Yukawa coupling matrix Y_{ai} is in general not diagonal $\Rightarrow$ in the decay of a mass-eigenstate sterile neutrino N_i any of the three charged leptons $e_a = e,\, \mu,\, \tau$ can be produced.

What are the conditions for the produced charged lepton state e_i to be a coherent superposition of the mass eigenstates e_a:

$$|e_i\rangle = [(Y^\dagger Y)_{ii}]^{-1/2} \sum_a Y^\dagger_{ia}\, |e_a\rangle \,,$$

and how long this state can maintain its coherence?

Extensions of the standard model?

Neglecting the masses of $\Phi^\pm$ and $l^\pm$ compared to the mass M_i of the sterile neutrino:

$$\Gamma_i^0 \simeq \alpha_i M_i\,, \qquad \text{where} \qquad \alpha_i \equiv \frac{(Y^\dagger Y)_{ii}}{16\pi}\,.$$

Coherent production condition:

$$2\sqrt{2}\,E\,\Gamma_i^0 \simeq 2\sqrt{2}\,(M_i/2)\,\alpha_i M_i > \max\{m_\mu^2 - m_e^2,\ m_\tau^2 - m_\mu^2\}\,,$$

or

$$\alpha_i > 2.2\,(\mathrm{GeV}/M_i)^2\,.$$

From $l_{\rm coh} = \sigma_x v_g/\Delta v_g$ the coherence length for the emitted charged lepton state:

$$l_{\rm coh} \simeq \frac{M_i^2}{2\Gamma_i^0(m_\tau^2 - m_\mu^2)} \simeq 3.1\times 10^{-15}\ \alpha_i^{-1}\,\frac{M_i}{\mathrm{GeV}}\ \ \mathrm{cm}\,.$$

$\Rightarrow$

Extensions of the standard model?

$$l_{\rm coh} < 1.4 \times 10^{-15}\ {\rm cm}\ (M_i/{\rm GeV})^3\,.$$

For N_i decays in flight the r.h.s. has to be multiplied by γ^3 $\Rightarrow$ $(M_i/{\rm GeV})^3$ has to be replaced by $(E_i/{\rm GeV})^3$.

The charged lepton state will maintain its coherence over the distance $\sim 1\ m$ if

$$E_i \gtrsim 400\ {\rm TeV} \quad \Rightarrow \quad (Y^\dagger Y)_{ii} \gtrsim 1.3 \times 10^{-11}\,.$$

If only e and μ are to be produced coherently, a milder lower limit on E_i results:

$$E_i \gtrsim 10\ {\rm TeV}\,, \qquad (Y^\dagger Y)_{ii} \gtrsim 8.5 \times 10^{-11}\,.$$

Extensions of the standard model?

If the condition for coherent creation of the charged lepton state is satisfied and this state is detected through the inverse decay process before it loses its coherence, it may exhibit oscillations: a mass eigenstate sterile neutrino N_j different from N_i can be produced in the detection process $\Rightarrow$ the state e_i has oscillated into e_j.

Charged leptons would be able to oscillate, leading to a non-zero probability of the emission or absorption of a different sterile neutrino mass eigenstate N_j in the processes $e_j^{\pm} + \Phi^{\mp} \to N_j$ or $e_j^{\pm} + N_j \to \Phi^{\pm}$.

$\Rightarrow$ The roles of neutrinos and charged leptons reversed compared to the usual situation because of sterile neutrinos being much heavier than the charged leptons.

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  - Automatically produces correctly normalized oscillation probability and clarifies the normalization prescription of QM approach
- $\Rightarrow$ the simplistic QM wave packet approach may need QFT-motivated modifications; however, once they have been done, one can still work within the QM framework without losing any essential physics content.