The Wigner Function on the Group $SO(2,1)$

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The Wigner quasi-probability distribution:

$$W(\vec{r},\vec{p}) = \int \frac{d\vec{R}}{(2\pi)^3}\ e^{-i\vec{p}\vec{R}}\psi^*\left(\vec{r}-\frac{1}{2}\vec{R}\right)\psi\left(\vec{r}+\frac{1}{2}\vec{R}\right).$$

Its properties:

$$\begin{aligned} &1.\ W^*(\vec{r},\ \vec{p}) = W(\vec{r},\ \vec{p})\ -\ (\text{reality})\ ,\\ &2.\ \int \mathrm{d}\vec{p}\ W(\vec{r},\vec{p}) = |\psi(\vec{r})|^2\ ,\\ &3.\ \int \mathrm{d}\vec{r}\ W(\vec{r},\vec{p}) = |\psi(\vec{p})|^2\ ,\\ &4.\ \int \mathrm{d}\vec{r}\,\mathrm{d}\vec{p}\ W(\vec{r},\vec{p}) = 1\ . \end{aligned} \tag{1}$$

An average value of any function $g(\vec{r},\ \vec{p})$:

$$\langle g\rangle = \int \mathrm{d}\vec{r}\,\mathrm{d}\vec{p}\ W(\vec{r},\vec{p})\ g(\vec{r},\ \vec{p})\ . \tag{2}$$

The Shapiro function

Wave function of a state with fixed value of the Casimir operator and projection of escape parameter onto the trnsverse vector satisfies the equations:

$$\left(d_\perp{}^2 - \frac{1}{q^2}\, L_3^2\right)\psi = b^2\psi\ , \quad (\vec{n}\vec{d}_\perp)\ \psi = \mathrm{const}\ \psi\ . \qquad (3)$$

It is (plane wave on the $SO_\mu(2,1)$ group):

$$\psi(u) = u_0\,(n\cdot u)^{-\frac{1}{2}+i\mu}\ ,$$
$$\mu = \sqrt{b^2q^2 - \frac{1}{4}}\ , \quad b^2q^2 \geq \frac{1}{4}. \qquad (4)$$

The conical variables:

$$u = (u_0,\ \vec{u}) = \left(\frac{q}{\sqrt{q^2 - \vec{q}_\perp{}^2}}\ , \ \frac{\vec{q}_\perp}{\sqrt{q^2 - \vec{q}_\perp{}^2}}\right)\ , \qquad (5)$$
$$u^2 = u_0^2 - \vec{u}\ ^2 = 1\ .$$

The small transverse momenta approximation

In the small transverse momenta approximation:

$$\psi(u) = e^{i\vec{\mu}\vec{u}} \; . \tag{6}$$

$$\vec{u} = u(cos\varphi,\; sin\varphi),\; 0 \le u < \infty,\; 0 \le \varphi \le 2\pi \; ,$$

$$\vec{\mu} = \mu(cos\psi,\; sin\psi),\; 0 \le \mu < \infty,\; 0 \le \psi \le 2\pi$$

An amplitude

$$F^{(\pm)}(\vec{u}) = \int e^{i\vec{u}\vec{\mu}} u^{(\pm)}(\vec{\mu})\, \mathrm{d}\vec{\mu} \tag{7}$$

The profile function on the $SO_\mu(2,1)$ group in the small transverse momenta approximation:

$$u^{(\pm)}(\vec{\mu}) = \frac{1}{(2\pi)^2} \int e^{-i\vec{u}\vec{\mu}} F^{(\pm)}(\vec{u})\, \mathrm{d}\vec{u} \tag{8}$$

Normalization of the amplitude $F^{(\pm)}(\vec{u})$:

$$\sigma_{el}^{(\pm)} = \int \left| F^{(\pm)}(\vec{u}) \right|^2 \, \mathrm{d}\vec{u} \; . \tag{9}$$

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The Wigner function

The $SO_\mu(2,1)$-transform of the correlator of scattering amplitudes:

$$W^{(\pm)}(\vec{u},\vec{\mu}) = \frac{1}{(2\pi)^2}\int e^{i\vec{\mu}\vec{v}} F^{(\pm)}\left(\vec{u}-\frac{\vec{v}}{2}\right)\ \bar{F}^{(\pm)}\left(\vec{u}+\frac{\vec{v}}{2}\right)\ \mathrm{d}\vec{v}\ .$$

Its properties:

$$\begin{aligned}
&1.\ \bar{W}^{(\pm)} = W^{(\pm)}\ \ -\ \ \text{(reality)}\ ,\\
&2.\ \int W^{(\pm)}(\vec{u},\vec{\mu})\ \mathrm{d}\vec{\mu} = \left|F^{(\pm)}(\vec{u})\right|^2 = \frac{\mathrm{d}\sigma_{el}^{(\pm)}}{\mathrm{d}\vec{u}}\ ,\\
&3.\ \int W^{(\pm)}(\vec{u},\vec{\mu})\ \mathrm{d}\vec{u} = (2\pi)^2\left|u^{(\pm)}(\vec{\mu})\right|^2 = \frac{\mathrm{d}\sigma_{el}^{(\pm)}}{\mathrm{d}\vec{\mu}}\ ,\\
&4.\ \int W^{(\pm)}(\vec{u},\vec{\mu})\ \mathrm{d}\vec{\mu}\ \mathrm{d}\vec{u} = \sigma_{el}^{(\pm)}\ .
\end{aligned} \tag{10}$$

An average value of any function $h(\vec{u},\ \vec{\mu})$:

$$\langle h(\vec{u},\ \vec{\mu})\rangle^{(\pm)} = \frac{1}{\sigma_{el}^{(\pm)}}\int h(\vec{u},\ \vec{\mu})\ W^{(\pm)}(\vec{u},\vec{\mu})\ \mathrm{d}\vec{\mu}\ \mathrm{d}\vec{u}\ . \tag{11}$$

Spatial escape parameter

The average value of squared minimum distance between scattered particles in elastic collisions:

$$\left\langle \mu^2 \right\rangle^{(\pm)} = \frac{1}{\sigma_{el}^{(\pm)}} \int \left| \nabla_{\vec{u}} F^{(\pm)}(\vec{u}) \right|^2 \, \mathrm{d}\vec{u} \; . \tag{12}$$

$$\left\langle \mu^2 \right\rangle = q^2 \left\langle b^2 \right\rangle - \frac{1}{4} \; . \tag{13}$$

Model 1: The one-particle exchange of a particle with the mass m in the t-channel

The amplitude in the terms of conical variables:

$$F^{(\epsilon)}(\vec{u}) = -\frac{g}{2p^2}\,\frac{u_0^{-3/2}}{1-\frac{\epsilon}{u_0}+\frac{m^2}{2p^2}}\ . \tag{14}$$

the g is the coupling constant, $\vec{p}$ is the momentum of initial particle.

The elastic cross-section:

$$\sigma_{el}^{(\epsilon)} = \frac{g^2}{4p^4}\,\frac{2\pi}{z_0(z_0-\epsilon)}\ ,\quad z_0 = 1+\frac{m^2}{2p^2}\ . \tag{15}$$

In the region of large energies of colliding particles ($m^2/p^2 \ll 1$):

$$\sqrt{\langle b^{\,2}\rangle^{(+)}} = \sqrt{\frac{2}{3}}\cdot\frac{\hbar}{mc}\ . \tag{16}$$

Model 2: The model of the elastic *pp*-scattering corresponding to exchange of the Pomeranchuk vacuum pole in the *t*-channel

The amplitude in the terms of conical variables:

$$F^{(\epsilon)}(\vec{u}) = \lambda e^{2p^2 B \frac{\epsilon}{u_0}} \cdot u_0^{-3/2} \;, \tag{17}$$

$$\lambda = ig(s)e^{-2p^2 B}, \qquad B = \alpha^{'}(0)\left(-i\frac{\pi}{2} + \ln\frac{s}{s_0} + \varkappa\right),$$

$\varkappa$ is a slope of the diffractive cone in differential cross-section of the *pp*-scattering; $s_0 \approx 100\ GeV^2$ determines the border of the high-energy regime area.

The elastic cross-section (forward hemisphere):

$$\sigma_{el}^{(+)}(p) = \frac{\pi\ g^2(s)}{2\ p^2\ \alpha'(0)\ \left(\ln\frac{s}{s_0} + \varkappa\right)} \;. \tag{18}$$

Model 2: The model of the elastic *pp*-scattering corresponding to exchange of the Pomeranchuk vacuum pole in the *t*-channel

In the high energy approximation:

$$\left\langle b^2\right\rangle^{(+)} = 2\alpha^{'}(0)\,\frac{\pi^2/4+\left(\varkappa+\ln\frac{s}{s_0}\right)^2}{\varkappa+\ln\frac{s}{s_0}}\ . \tag{19}$$

At $s\approx s_0$:

$$\sqrt{\langle b^2\rangle^{(+)}}\approx(1.26\ \div\ 1.46)\ fm\ . \tag{20}$$

At the SPS-collider energy $\sqrt{s}\approx 540\ GeV$:

$$\sqrt{\langle b^2\rangle^{(+)}}\approx(1.64\ \div\ 1.8)\ fm\ . \tag{21}$$

At the LHC energy $\sqrt{s}\approx 14\ TeV$:

$$\sqrt{\langle b^2\rangle^{(+)}}\approx(1.88\ \div\ 2.02)\ fm\ . \tag{22}$$

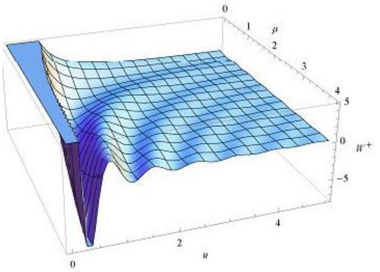
The Wigner function for the model of the one-particle exchange

Explicit form of the Wigner function for the model of the one-particle exchange:

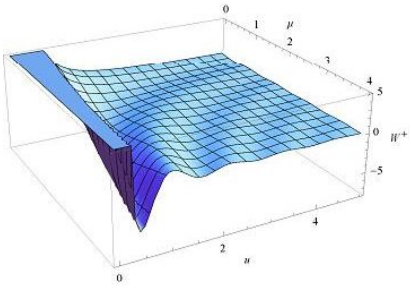
$$W^{(+)}(\vec{u},\ \vec{\mu}) =$$

$$= \frac{4\ g^2}{2\pi\ p^4} \int\limits_0^1 \mathrm{d}\alpha\ \frac{|\vec{\mu}|\ cos\left[2\vec{\mu}\vec{u}(1-2\alpha)\right]}{\sqrt{4\alpha(1-\alpha)u^2 + \frac{m^2}{p^2}}}\ K_1\left(2|\vec{\mu}|\sqrt{4\alpha(1-\alpha)u^2 + \frac{m^2}{p^2}}\right) \tag{23}$$

The Wigner function for the model of the one-particle exchange



The Wigner function for the model of the one-particle exchange. Dependence on $|\vec{\mu}|$ and $|\vec{u}|$ at the $\varphi = 0$ is shown.



The Wigner function for the model of the one-particle exchange. Dependence on $|\vec{\mu}|$ and $|\vec{u}|$ at the $\varphi = \pi/4$ is shown.

Thank you for your attention!