QCD

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Quark field

q^i colour $i \in [1, N_c]$

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$$q^i \to U^i{}_j q^j \quad \text{or} \quad q \to Uq$$
$$U^+U = 1 \qquad \det U = 1$$

Fundamental representation of $SU(N_c)$

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Fundamental representation of $SU(N_c)$

$$\bar{q}_i = (q^i)^+ \gamma^0$$
$$\bar{q}_i \to U_i{}^j \bar{q}_j \quad \text{or} \quad \bar{q} \to \bar{q} U^+ \quad \text{where} \quad U_i{}^j = (U^i{}_j)^*$$

Conjugated fundamental representation

Invariant tensors

Meson

$$\bar{q}q' \to \bar{q}U^{+}Uq' = \bar{q}q'$$

$$\delta^i_j \to \delta^{i'}_{j'} U^i{}_{i'} U_j{}^{j'} = U^i{}_k U_j{}^k = \delta^i_j$$

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Baryon $(N_c = 3)$

$$\varepsilon_{ijk}q_1^i q_2^j q_3^k \to \varepsilon_{ijk}U^i{}_{i'}U^j{}_{j'}U^k{}_{k'}q_1^{i'}q_2^{j'}q_3^{k'} = \det U \cdot \varepsilon_{i'j'k'}q_1^{i'}q_2^{j'}q_3^{k'}$$
$$= \varepsilon_{ijk}q_1^i q_2^j q_3^k$$
$$\varepsilon_{ijk} \to \varepsilon_{i'j'k'}U_i{}^{i'}U_j{}^{j'}U_k{}^{k'} = \det U^{+} \cdot \varepsilon_{ijk} = \varepsilon_{ijk}$$

Invariant tensors

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Baryon $(N_c = 3)$

$$\varepsilon_{ijk} q_1^i q_2^j q_3^k \to \varepsilon_{ijk} {U^i}_{i'} {U^j}_{j'} {U^k}_{k'} q_1^{i'} q_2^{j'} q_3^{k'} = \det U \cdot \varepsilon_{i'j'k'} q_1^{i'} q_2^{j'} q_3^{k'}$$
$$= \varepsilon_{ijk} q_1^i q_2^j q_3^k$$
$$\varepsilon_{ijk} \to \varepsilon_{i'j'k'} {U_i}^{i'} {U_j}^{j'} {U_k}^{k'} = \det U^{+} \cdot \varepsilon_{ijk} = \varepsilon_{ijk}$$

Antibaryon $(N_c = 3)$

$$\varepsilon^{ijk} \bar{q}_{1i} \bar{q}_{2j} \bar{q}_{3k} \to \varepsilon^{ijk} \bar{q}_{1i} \bar{q}_{2j} \bar{q}_{3k}$$
$$\varepsilon^{ijk} \to \varepsilon^{ijk}$$

Infinitesimal transformations

$$U = 1 + i\alpha^a t^a$$

t^a — generators of the fundamental representation

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$$\det U = 1 + i\alpha^a \operatorname{Tr} t^a = 1 \quad \Rightarrow \quad \operatorname{Tr} t^a = 0$$

$$\operatorname{Tr} t^a t^b = T_F \delta^{ab}$$

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Space	Dimension
hermitian matrices	N_c^2
traceless hermitian matrices	$N_c^2 - 1$

$N_c^2 - 1$ generators t^a form a basis

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$N_c^2 - 1$ generators t^a form a basis
$[t^a, t^b]$ is antihermitian and traceless

$$[t^a, t^b] = i f^{abc} t^c \qquad i f^{abc} = \frac{1}{T_F} \operatorname{Tr}[t^a, t^b] t^c$$

f^{abc} — structure constants

Adjoint representation

$$A^a = \bar{q} t^a q'$$

$$A^a \to \bar{q} U^+ t^a U q' = U^{ab} A^b$$

$$U^+ t^a U = U^{ab} t^b$$

$$U^{ab} = \frac{1}{T_F} \operatorname{Tr} U^+ t^a U t^b$$

Adjoint representation

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U^+ t^a U &= U^{ab} t^b \\
U^{ab} &= \frac{1}{T_F} \operatorname{Tr} U^+ t^a U t^b
\end{aligned}
$$

Invariant tensor $(t^a)^i{}_j$

$$(t^a)^i{}_j \to U^{ab} U^i{}_{i'} U_j{}^{j'} (t^b)^{i'}{}_{j'} = (t^a)^i{}_j$$

Infinitesimal transformations

$$A^a \to \bar{q}(1 - i\alpha^c t^c) t^a (1 + i\alpha^c t^c) q' = \bar{q}(t^a + i\alpha^c i f^{acb} t^b) q'$$

$$U^{ab} = \delta^{ab} + i\alpha^c (t^c)^{ab}$$

$$(t^c)^{ab} = i f^{acb}$$

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$$U^{ab} = \delta^{ab} + i\alpha^c (t^c)^{ab}$$
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$$(t^a)^{dc}(t^b)^{ce} - (t^b)^{dc}(t^a)^{ce} = i f^{abc}(t^c)^{de}$$

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$$(t^a)^{dc}(t^b)^{ce} - (t^b)^{dc}(t^a)^{ce} = i f^{abc}(t^c)^{de}$$

Jacobi identity

$$[t^a, [t^b, t^d]] + [t^b, [t^d, t^a]] + [t^d, [t^a, t^b]] = 0$$
$$= \left(i f^{bdc} i f^{ace} + i f^{dac} i f^{bce} + i f^{abc} i f^{dce}\right) t^e$$

Gluon field

Free quark field

$$L = \bar{q}(i\gamma^\mu \partial_\mu - m)q$$

Invariant with respect to $q(x) \to Uq(x)$

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Invariant with respect to $q(x) \to Uq(x)$
How to make it invariant for $q(x) \to U(x)q(x)$?

$$\partial_\mu q \;\Rightarrow\; D_\mu q$$

$$D_\mu q = (\partial_\mu - igA_\mu)q \qquad A_\mu = A_\mu^a t^a$$

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$$\partial_\mu q \;\Rightarrow\; D_\mu q$$
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$$q \to q' = Uq \qquad A_\mu \to A'_\mu \qquad D_\mu q \to D'_\mu q' = UD_\mu q$$
$$(\partial_\mu - igA'_\mu)Uq = U(\partial_\mu - igA_\mu)q$$
$$\partial_\mu U - igA'_\mu U = -igUA_\mu$$
$$A'_\mu = UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}$$

Gluon field strength

Infinitesimal tramsformation

$$q(x) \to q'(x) = (1 + i\alpha^a(x)t^a)a(x)$$

$$A^a_\mu(x) \to A'^a_\mu(x) = A^a_\mu(x) + \frac{1}{g} D^{ab}_\mu \alpha^b(x)$$

$$D^{ab}_\mu = \delta^{ab}\partial_\mu - ig(t^c)^{ab} A^c_\mu$$

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$$D_\mu^{ab} = \delta^{ab}\partial_\mu - ig(t^c)^{ab}A_\mu^c$$

$$\begin{aligned}
&[D_\mu, D_\nu]q \\
&= \partial_\mu\partial_\nu q - ig(\partial_\mu A_\nu)q - igA_\nu\partial_\mu q - igA_\mu\partial_\nu q - g^2 A_\mu A_\nu q \\
&- \partial_\nu\partial_\mu q + ig(\partial_\nu A_\mu)q + igA_\mu\partial_\nu q + igA_\nu\partial_\mu q + g^2 A_\nu A_\mu q \\
&= -igG_{\mu\nu}q \\
&G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] = G_{\mu\nu}^a t^a \\
&G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c
\end{aligned}$$

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$$D^{ab}_\mu = \delta^{ab}\partial_\mu - ig(t^c)^{ab}A^c_\mu$$

$$\begin{aligned}
&[D_\mu, D_\nu]q \\
&= \partial_\mu\partial_\nu q - ig(\partial_\mu A_\nu)q - igA_\nu\partial_\mu q - igA_\mu\partial_\nu q - g^2 A_\mu A_\nu q \\
&- \partial_\nu\partial_\mu q + ig(\partial_\nu A_\mu)q + igA_\mu\partial_\nu q + igA_\nu\partial_\mu q + g^2 A_\nu A_\mu q \\
&= -igG_{\mu\nu}q \\
&G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] = G^a_{\mu\nu}t^a \\
&G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu
\end{aligned}$$

$$G_{\mu\nu} \to UG_{\mu\nu}U^{-1} \qquad G^a_{\mu\nu} \to U^{ab}G^b_{\mu\nu}$$

QCD Lagrangian

$$L = L_q + L_A$$

$$L_q = \sum_f \bar{q}_f (i\gamma^\mu D_\mu + m_f) q_f$$

$$L_A = -\frac{1}{2} \operatorname{Tr} G_{\mu\nu} G^{\mu\nu} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}$$

Flavour symmetries

$U(1)$: $q_f \to e^{i\alpha} q_f \approx (1 + i\alpha) q_f$
$\Rightarrow$ baryon number conservation

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$\Rightarrow$ baryon number conservation

$m_f = m$

$$q_f \to U_{ff'} q_{f'} \qquad U^+ U = 1$$

$U(n_f) = U(1) \times SU(n_f)$

$$U = e^{i\alpha} U_0 \qquad \det U_0 = 1$$

Infinitesimal transformation

$$U = 1 + i\alpha + i\alpha^a \tau^a \qquad \mathrm{Tr}\, \tau^a = 0$$

Chiral symmetries

$m_f = 0$

$$q_f = q_{Lf} + q_{Rf} \qquad q_{L,R} = \frac{1 \pm \gamma_5}{2} q \qquad \gamma_5 q_{L,R} = \pm q_{L,R}$$

$$L = \sum_f \bar{q}_{Lf} i\gamma^\mu D_\mu q_{Lf} + \sum_f \bar{q}_{Rf} i\gamma^\mu D_\mu q_{Rf}$$

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$$U(n_f)_L \times U(n_f)_R$$

$$q_L \to (1 + i\alpha_L + i\alpha_L^a \tau^a) q_L$$
$$q_R \to (1 + i\alpha_R + i\alpha_R^a \tau^a) q_L$$

Chiral symmetries

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$$U(n_f)_L \times U(n_f)_R$$

$$q_L \to (1 + i\alpha_L + i\alpha_L^a \tau^a) q_L$$
$$q_R \to (1 + i\alpha_R + i\alpha_R^a \tau^a) q_L$$

$$U(n_f)_V \times U(n_f)_A$$

$$q \to (1 + i\alpha_V + i\alpha_V^a \tau^a + i\alpha_A \gamma_5 + i\alpha_A^a \tau^a \gamma_5) q$$

Conformal symmetry

Scale symmetry

$$x^\mu \to \lambda x^\mu \qquad A^\mu \to \lambda^{-1} A^\mu \qquad q \to \lambda^{-3/2} q$$

no dimensional parameters in the Lagrangian

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$$x^\mu \to \frac{x^\mu}{x^2}$$

Conformal symmetry

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no dimensional parameters in the Lagrangian

Inversion

$$x^\mu \to \frac{x^\mu}{x^2}$$

Special conformal transformation

inversion $\Rightarrow$ translation $\Rightarrow$ inversion

$$x^\mu \to \frac{x^\mu + a^\mu x^2}{1 + 2a \cdot x + a^2 x^2}$$

Symmetries

Group	Classical	Quantum
translation Lorentz		
conformal		anomaly
$SU(N_c)$	local	
$U(1)$ $SU(n_f)$		
$U(1)_A$ $SU(n_f)_A$		anomaly spontaneously broken
P C T	discrete	

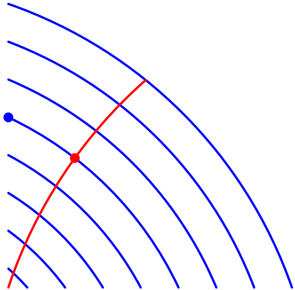
Functional integral

$$< T\{O(x), O(y)\} > = \frac{\int \prod_{x,a,\mu} dA_\mu^a(x)\, e^{i\int L\, d^4x}\, O(x)\, O(y)}{\int \prod_{x,a,\mu} dA_\mu^a(x)\, e^{i\int L\, d^4x}}$$

$$= \frac{1}{i^2}\, \frac{\delta^2 Z[j]}{\delta j(x)\, \delta j(y)}\bigg|_{j=0}$$

$$Z[j] = \int \prod_{x,a,\mu} dA_\mu^a(x)\, e^{i\int (L+jO)\, d^4x}$$

Functional integral for gauge fields



Orbits of the gauge group $A \to A^U$

Gauge $G^a(A^U(x)) = 0$ — unique solution $U(x)$ for any given $A(x)$

e.g. Lorentz gauge $G^a(A(x)) = \partial^\mu A^a_\mu(x)$

The surface $G = 0$ intersects any orbit at 1 point

Faddeev–Popov determinant

$$\Delta^{-1}[A] = \int \prod_x dU(x) \prod_{x,a} \delta(G^a(A^U(x)))$$

Invariant measure $d(U_0 U) = dU$

for infinitesimal transformations $dU = \prod_a d\alpha^a$

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Near the surface $G(A^{U_0}) = 0$:

$$\delta G(A(x)) = \hat{M}\alpha(x)$$

$$\Delta^{-1}[A] = \int \prod_x \alpha(x)\delta(\hat{M}\alpha(x)) = 1/\det \hat{M}$$

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Lorentz gauge $G^a(A(x)) = \partial^\mu A^a_\mu$

$$\delta G^a(x) = \frac{1}{g}\partial^\mu D^{ab}_\mu \alpha^b(x) \;\Rightarrow\; \hat{M} = \frac{1}{g}\partial^\mu D^{ab}_\mu$$

Faddeev–Popov determinant is gauge invariant

$$\Delta^{-1}[A^{U_0}] = \int \prod_x dU \prod_{x,a} \delta(G^a(A^{U_0 U}(x)))$$

$$= \int \prod_x D(U_0 U) \prod_{x,a} \delta(G^a(A^{U_0 U}(x))) = \Delta^{-1}[A]$$

Functional integral in a fixed gauge

$$
\begin{aligned}
Z[J] &= \int \prod_x dA(x)\, e^{iS[A]} \\
&= \int \prod_x dU(x) \prod_x dA(x)\, \Delta[A] \prod_x \delta(G(A^U(x)))\, e^{iS[A]} \\
&= \left(\prod_x \int dU \right) \times \int \prod_x dA(x)\, \Delta[A] \prod_x \delta(G(A(x)))\, e^{iS[A]}
\end{aligned}
$$

Gauss integrals

Bosonic

$$\int dz^* \, dz \, e^{-a z^* z} \sim \frac{1}{a}$$

$$\int \prod_i dz_i^* \, dz_i \, e^{-M_{ij} z_i^* z_j} \sim \frac{1}{\det M}$$

Gauss integrals

Bosonic

$$\int dz^*\, dz\, e^{-az^*z} \sim \frac{1}{a}$$

$$\int \prod_i dz_i^*\, dz_i\, e^{-M_{ij} z_i^* z_j} \sim \frac{1}{\det M}$$

Fermionic

$$\int dc^*\, dc\, e^{-ac^*c} \sim a$$

$$\int \prod_i dc_i^*\, dc_i\, e^{-M_{ij} c_i^* c_j} \sim \det M$$

Ghost field

Generalized Lorentz gauge $G^a(A(x)) = \partial^\mu A^a_\mu(x) - \omega^a(x)$

$$\Delta = \det \hat{M} \sim \det \partial^\mu D^{ab}_\mu = \int \prod_{x,a} d\bar{c}^a(x)\, dc^a(x)\, e^{i\int L_c d^4x}$$

$$L_c = -\bar{c}^a \partial^\mu D^{ab}_\mu c^b \Rightarrow (\partial^\mu \bar{c}^a) D^{ab}_\mu c^b$$

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$$Z[J] = \int \prod_{x,a} dA^a(x)\, d\bar{c}^a(x)\, dc^a(x) \times \prod_{x,a} \delta(\partial^\mu A^a_\mu(x) - \omega^a(x))\, e^{i\int (L_A + L_c + JO) d^4x}$$

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$$\begin{aligned} Z[J] = & \int \prod_{x,a} dA^a(x)\, d\bar{c}^a(x)\, dc^a(x) \\ & \times \prod_{x,a} \delta(\partial^\mu A_\mu^a(x) - \omega^a(x))\, e^{i\int (L_A + L_c + JO) d^4x} \\ & \times \prod_{x,a} d\omega^a(x)\, e^{-\frac{i}{2a}\int \omega^a(x)\omega^a(x)\, d^4x} \end{aligned}$$

QCD generating functional

$$Z[J] = \int \prod_{x,a} dA^a(x)\, d\bar{c}^a(x)\, dc^a(x)\, e^{i\int (L+JO) d^4x}$$

$$L = L_A + L_c + L_f$$

$$L_A = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}$$

$$L_c = (\partial^\mu \bar{c}^a) D^{ab}_\mu c^b$$

$$L_f = -\frac{1}{2a} \left(\partial^\mu A^a_\mu\right)^2$$

QCD generating functional

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$$L_f = -\frac{1}{2a} \left(\partial^\mu A^a_\mu\right)^2$$

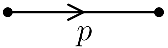
QED: ghost does not interact

BRST symmetry

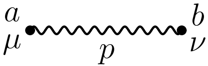
$$
\begin{aligned}
\delta A_\mu^a &= \lambda^+ D_\mu^{ab} c^b \\
\delta \bar{c}^a &= -\frac{1}{a} \lambda^+ \partial^\mu A_\mu^a \\
\delta c^a &= -\frac{g}{2} f^{abc} \lambda^+ c^b c^c
\end{aligned}
$$

λ — a constant infinitesimal fermionic parameter

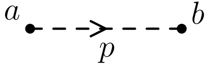
Feynman rules



$= iS_0(p) \qquad S_0(p) = \frac{1}{\not{p}} = \frac{\not{p}}{p^2}$

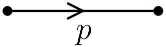


$= -i\delta^{ab} D^0_{\mu\nu}(p)$

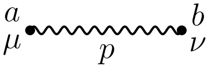


$= i\delta^{ab} G_0(p) \qquad G_0(p) = \frac{1}{p^2}$

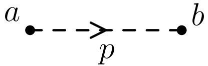
Feynman rules



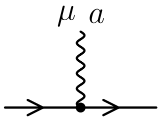
$$= iS_0(p) \qquad S_0(p) = \frac{1}{\not p} = \frac{\not p}{p^2}$$



$$= -i\delta^{ab} D^0_{\mu\nu}(p)$$

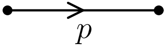


$$= i\delta^{ab} G_0(p) \qquad G_0(p) = \tfrac{1}{p^2}$$

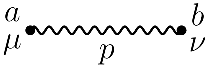


$$= t^a \times i g_0 \gamma^\mu$$

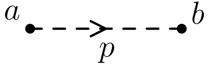
Feynman rules



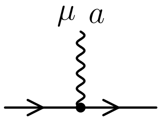
$$= iS_0(p) \qquad S_0(p) = \frac{1}{\not{p}} = \frac{\not{p}}{p^2}$$



$$= -i\delta^{ab} D^0_{\mu\nu}(p)$$



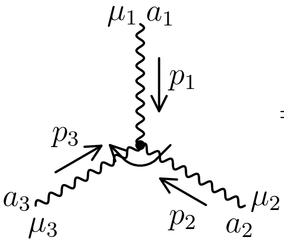
$$= i\delta^{ab} G_0(p) \qquad G_0(p) = \tfrac{1}{p^2}$$



$$= t^a \times i g_0 \gamma^\mu$$

$$D^0_{\mu\nu}(p) = \frac{1}{p^2}\left[g_{\mu\nu} - (1 - a_0)\frac{p_\mu p_\nu}{p^2}\right]$$

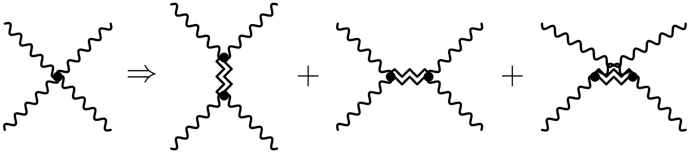
Feynman rules

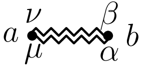


$$= if^{a_1 a_2 a_3} \times i g_0 V^{\mu_1 \mu_2 \mu_3}(p_1, p_2, p_3)$$

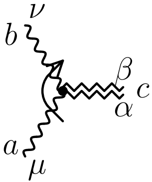
$$V^{\mu_1 \mu_2 \mu_3}(p_1, p_2, p_3) =$$
$$(p_3 - p_2)^{\mu_1} g^{\mu_2 \mu_3} + (p_1 - p_3)^{\mu_2} g^{\mu_3 \mu_1} + (p_2 - p_1)^{\mu_3} g^{\mu_1 \mu_2}$$

Feynman rules



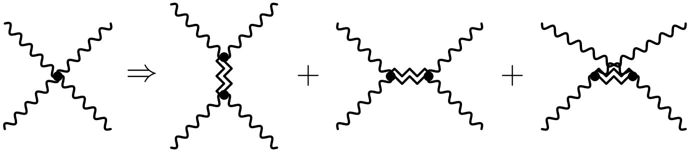


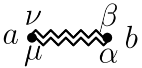
$$= i\delta^{ab}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})$$



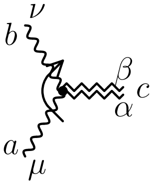
$$= if^{abc} \times g_0 g^{\mu\alpha}g^{\nu\beta}$$

Feynman rules

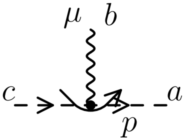




$$= i\delta^{ab}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})$$

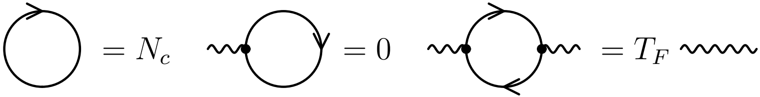


$$= if^{abc} \times g_0 g^{\mu\alpha}g^{\nu\beta}$$



$$= if^{abc} \times ig_0 p^{\mu}$$

Colour factors



$\mathrm{Tr}\, 1 = N_c$ $\quad$ $\mathrm{Tr}\, t^a = 0$ $\quad$ $\mathrm{Tr}\, t^a t^b = T_F \delta^{ab}$

$$T_F = \frac{1}{2}$$

Cvitanović algorithm

$$(t^a)^i{}_j (t^a)^k{}_l = a\left[\delta^i_l \delta^k_j - b\delta^i_j \delta^k_l\right]$$

Cvitanović algorithm

$$(t^a)^i{}_j (t^a)^k{}_l = a\left[\delta^i_l \delta^k_j - b\delta^i_j \delta^k_l\right]$$

Multiply by δ^j_i

$$(t^a)^i{}_i (t^a)^k{}_l = 0 = a\left[\delta^k_l - bN_c \delta^k_l\right]$$

$$b = \frac{1}{N_c}$$

Cvitanović algorithm

$$(t^a)^i{}_j (t^a)^k{}_l = a \left[\delta^i_l \delta^k_j - b \delta^i_j \delta^k_l \right]$$

Multiply by δ^j_i

$$(t^a)^i{}_i (t^a)^k{}_l = 0 = a \left[\delta^k_l - b N_c \delta^k_l \right]$$

$$b = \frac{1}{N_c}$$

Multiply by $(t^b)^j{}_i$

$$(t^b)^j{}_i (t^a)^i{}_j (t^a)^k{}_l = T_F (t^b)^k{}_l = a \left[(t^b)^k{}_l - \frac{1}{N_c} (t^b)^i{}_i \delta^k_l \right]$$

$$a = T_F$$

Cvitanović algorithm

$$(t^a)^i{}_j (t^a)^k{}_l = a\left[\delta^i_l \delta^k_j - b\delta^i_j \delta^k_l\right]$$

Multiply by δ^j_i

$$(t^a)^i{}_i (t^a)^k{}_l = 0 = a\left[\delta^k_l - bN_c\delta^k_l\right]$$

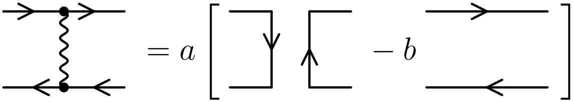
$$b = \frac{1}{N_c}$$

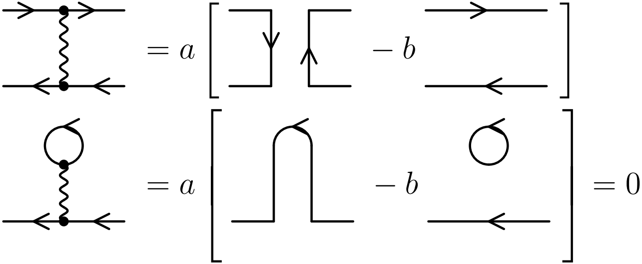
Multiply by $(t^b)^j{}_i$

$$(t^b)^j{}_i (t^a)^i{}_j (t^a)^k{}_l = T_F (t^b)^k{}_l = a\left[(t^b)^k{}_l - \frac{1}{N_c}(t^b)^i{}_i \delta^k_l\right]$$

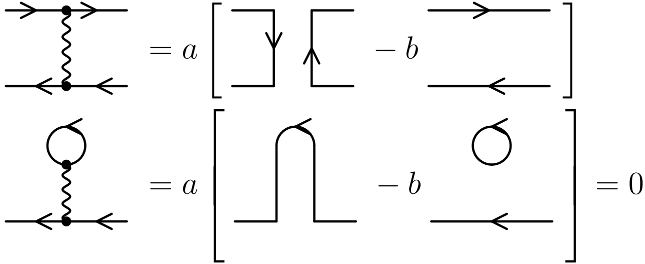
$$a = T_F$$

$$(t^a)^i{}_j (t^a)^k{}_l = T_F\left[\delta^i_l \delta^k_j - \frac{1}{N_c}\delta^i_j \delta^k_l\right]$$

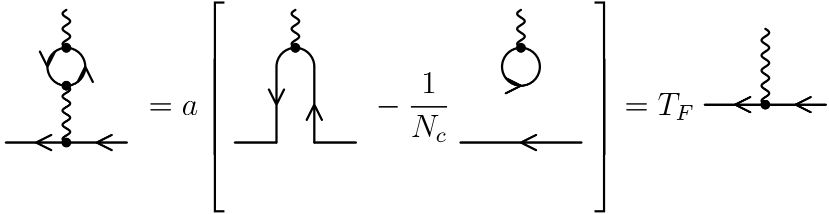

= a
− b



$$b = \frac{1}{N_c}$$

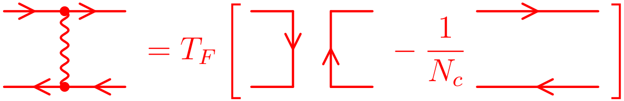


$$b = \frac{1}{N_c}$$

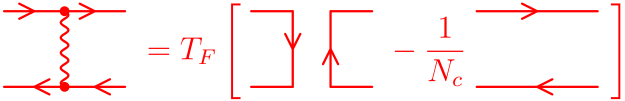


$$a = T_F$$

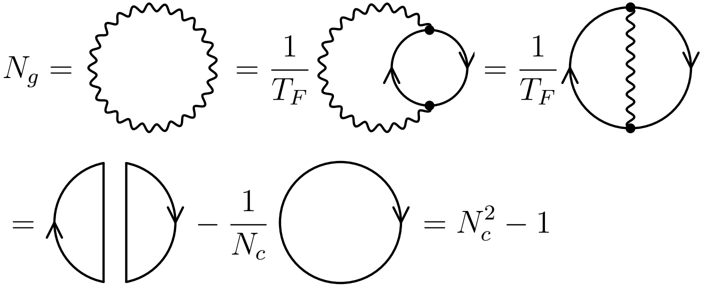
Cvitanović algorithm



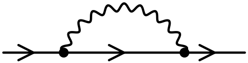
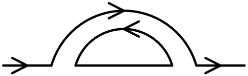
Cvitanović algorithm



Counting gluons



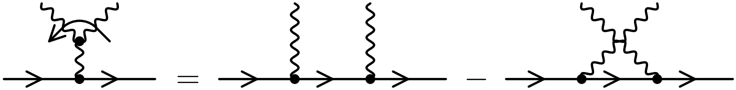
C_F

 $= T_F \Big[$  $- \frac{1}{N_c}$  $\Big]$

$= T_F \left(N_c - \frac{1}{N_c} \right)$ 

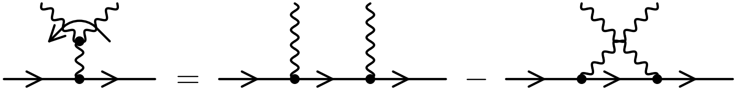
$$t^a t^a = C_F \qquad C_F = T_F \left(N_c - \frac{1}{N_c} \right)$$

3-gluon vertex

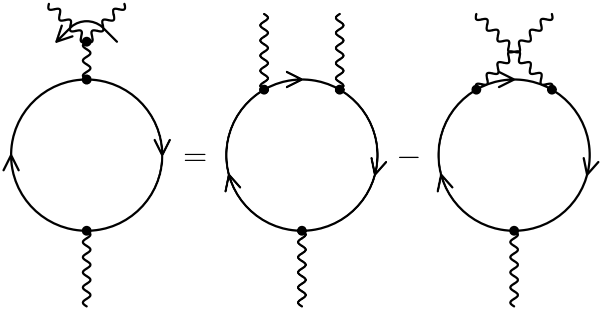


$$[t^a, t^b] = i f^{abc} t^c$$

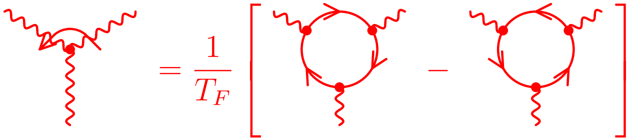
3-gluon vertex



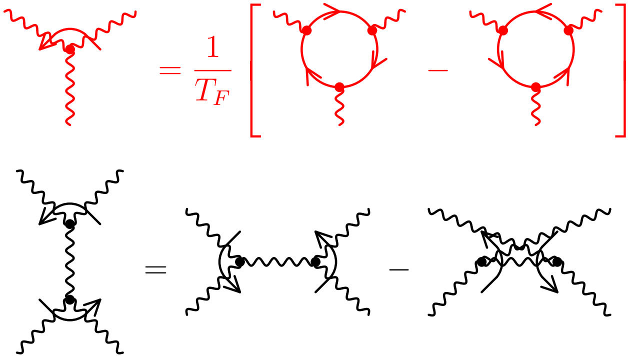
$[t^a, t^b] = i f^{abc} t^c$



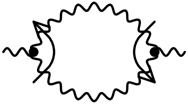
3-gluon vertex

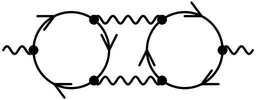
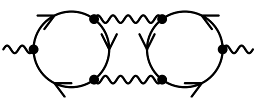


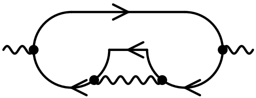
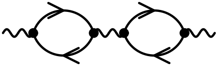
3-gluon vertex

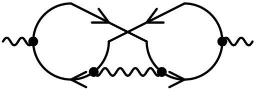
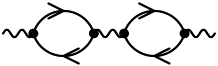


C_A

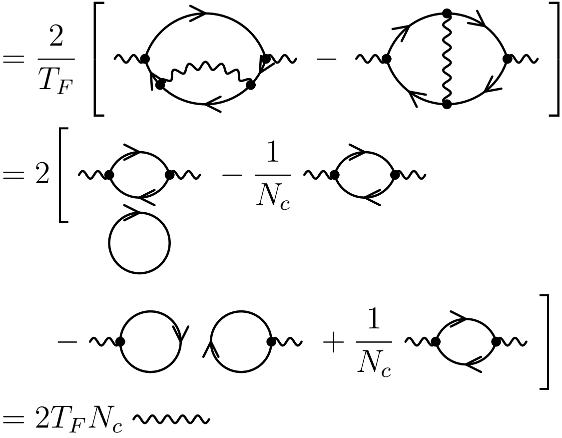


$= \frac{2}{T_F^2} \Bigg[$  $-$  $\Bigg]$

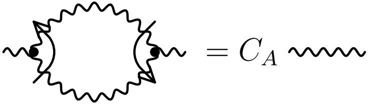
$= \frac{2}{T_F} \Bigg[$  $- \frac{1}{N_c}$ 

$-$  $+ \frac{1}{N_c}$  $\Bigg]$

C_A

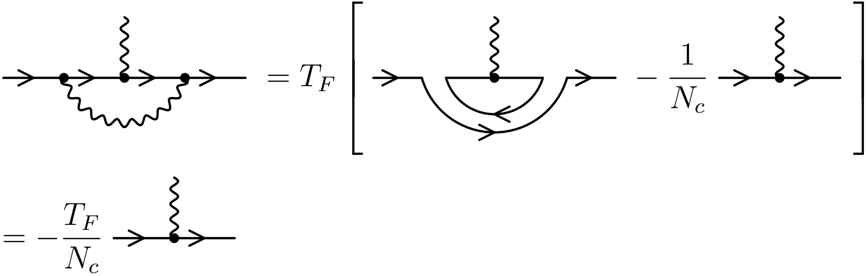


C_A



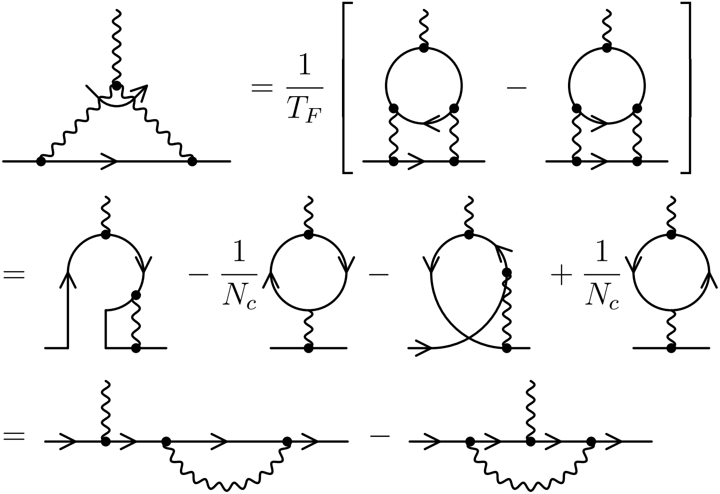
$$if^{acd} if^{bdc} = C_A \delta^{ab} \qquad C_A = 2 T_F N_c$$

Example 1

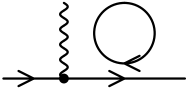
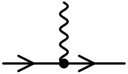


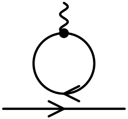
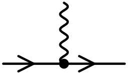
$$t^a t^b t^a = -\frac{T_F}{N_c} t^b \qquad -\frac{T_F}{N_c} = C_F - \frac{C_A}{2}$$

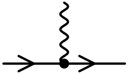
Example 2



Example 2

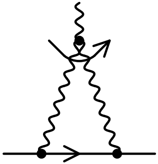
$= T_F \Big[$  $- \frac{1}{N_c}$ 

$-$  $+ \frac{1}{N_c}$  $\Big]$

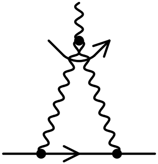
$= T_F N_c$ 

$$i f^{abc} t^b t^a = \frac{C_A}{2} t^c$$

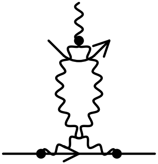
Shorter solution



$= \frac{1}{2}$ [

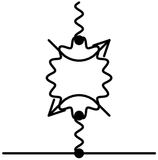


$-$

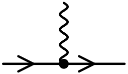


]

$= \frac{1}{2}$

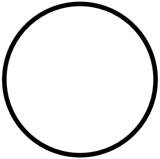


$= \frac{C_A}{2}$



1-loop massive vacuum diagram

We are going to live in $d = 4 - 2\varepsilon$ dimensional space-time

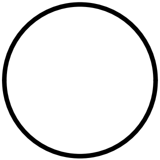


$$\int \frac{d^d k}{D^n} = i\pi^{d/2} m^{d-2n} V(n)$$

$$D = m^2 - k^2 - i0$$

1-loop massive vacuum diagram

We are going to live in $d = 4 - 2\varepsilon$ dimensional space-time



$$\int \frac{d^d k}{D^n} = i\pi^{d/2} m^{d-2n} V(n)$$

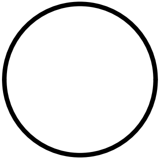
$$D = m^2 - k^2 - i0$$

Wick rotation $k_0 = ik_0$, $k^2 = -k^2$

$$\int \frac{d^d k}{(k^2 + 1)^n} = \pi^{d/2} V(n)$$

1-loop massive vacuum diagram

We are going to live in $d = 4 - 2\varepsilon$ dimensional space-time



$$\int \frac{d^d k}{D^n} = i\pi^{d/2} m^{d-2n} V(n)$$

$$D = m^2 - k^2 - i0$$

Wick rotation $k_0 = i k_0$, $k^2 = -k^2$

$$\int \frac{d^d k}{(k^2 + 1)^n} = \pi^{d/2} V(n)$$

α parametrization

$$\frac{1}{a^n} = \frac{1}{\Gamma(n)} \int_0^\infty e^{-a\alpha} \alpha^{n-1} d\alpha$$

1-loop massive vacuum diagram

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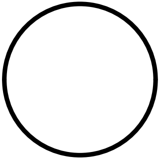

$$\int \frac{d^d k}{D^n} = i\pi^{d/2} m^{d-2n} V(n)$$

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$$V(n) = \frac{\pi^{-d/2}}{\Gamma(n)} \int e^{-\alpha(k^2+1)} \alpha^{n-1} d\alpha \, d^d k$$

1-loop massive vacuum diagram

$$\int e^{-\alpha k^2} d^d k = \left[\int_{-\infty}^{+\infty} e^{-\alpha k_x^2} dk_x \right]^d = \left(\frac{\pi}{\alpha} \right)^{d/2}$$

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$$V(n) = \frac{\Gamma(-d/2+n)}{\Gamma(n)}$$

For integer n, proportional to

$$V_1 = \frac{4}{(d-2)(d-4)} \Gamma(1+\varepsilon)$$

d-dimensional solid angle

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$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

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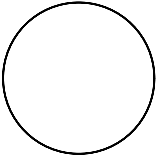
$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

$$\Omega_1 = 2 \qquad \Omega_2 = 2\pi \qquad \Omega_3 = 4\pi \qquad \Omega_4 = 2\pi^2 \qquad \cdots$$

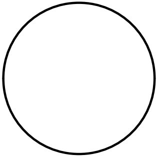
Massless vacuum diagrams



$$\int \frac{d^d k}{(-k^2 - i0)^n} = 0$$

by dimensionality (argument fails at $n = d/2$)

Massless vacuum diagrams



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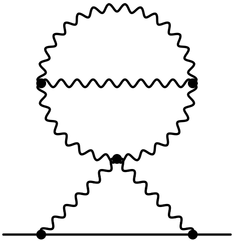

$$= 0$$

Feynman parametrization

α parametrization

$$\frac{1}{a_1^{n_1} a_2^{n_2}} = \frac{1}{\Gamma(n_1)\Gamma(n_2)} \int e^{-a_1\alpha_1 - a_2\alpha_2} \alpha_1^{n_1-1} \alpha_2^{n_2-1} d\alpha_1 \, d\alpha_2$$

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Substitution $\alpha_1 = \eta x$, $\alpha_2 = \eta(1-x)$

$$\frac{1}{a_1^{n_1} a_2^{n_2}} = \frac{\Gamma(n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 \frac{x^{n_1-1}(1-x)^{n_2-1}dx}{[a_1 x + a_2(1-x)]^{n_1+n_2}}$$

Feynman parametrization

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Feynman parametrization

$$\frac{1}{a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}} = \frac{1}{\Gamma(n_1)\Gamma(n_2)\cdots\Gamma(n_k)}$$

$$\int e^{-a_1\alpha_1 - a_2\alpha_2 \cdots - a_k\alpha_k} \alpha_1^{n_1-1} \alpha_2^{n_2-1} \cdots \alpha_k^{n_k-1} d\alpha_1\, d\alpha_2 \cdots d\alpha_k$$

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$$\times\, \delta(\alpha_1 + \alpha_2 \cdots + \alpha_l - \eta) d\eta \qquad (1 \leq l \leq k)$$

Feynman parametrization

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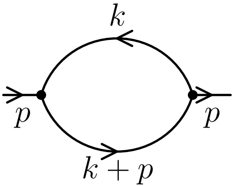
$$\times\, \delta(\alpha_1 + \alpha_2 \cdots + \alpha_l - \eta) d\eta \qquad (1 \le l \le k)$$

$$\alpha_i = \eta x_i$$

$$\frac{1}{a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}} = \frac{\Gamma(n_1 + n_2 \cdots + n_k)}{\Gamma(n_1)\Gamma(n_2)\cdots\Gamma(n_k)}$$

$$\int \frac{\delta(x_1 + x_2 \cdots + x_l - 1) x_1^{n_1-1} x_2^{n_2-1} \cdots x_k^{n_k-1} dx_1\, dx_2 \cdots dx_k}{[a_1x_1 + a_2x_2 \cdots + a_kx_k]^{n_1+n_2\cdots+n_k}}$$

1-loop massless propagator diagram

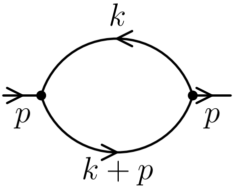


$$\int \frac{d^d k}{D_1^{n_1} D_2^{n_2}} = i\pi^{d/2}(-p^2)^{d/2-n_1-n_2} G(n_1, n_2)$$

$$D_1 = -(k+p)^2 \qquad D_2 = -k^2$$

Vanishes for integer $n_1 \leq 0$ or $n_2 \leq 0$

1-loop massless propagator diagram



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Vanishes for integer $n_1 \leq 0$ or $n_2 \leq 0$
Wick rotation, α parametrization

$$G(n_1, n_2) = \frac{\pi^{-d/2}}{\Gamma(n_1)\Gamma(n_2)}$$
$$\times \int e^{-\alpha_1 (k+p)^2 - \alpha_2 k^2} \alpha_1^{n_1-1} \alpha_2^{n_2-1} d\alpha_1 \, d\alpha_2 \, d^d k$$

1-loop massless propagator diagram

Shift $k' = k + \frac{\alpha_1}{\alpha_1 + \alpha_2} p$

$$G(n_1, n_2) = \frac{\pi^{-d/2}}{\Gamma(n_1)\Gamma(n_2)} \int \exp\left[-\frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2}\right] \alpha_1^{n_1 - 1} \alpha_2^{n_2 - 1} d\alpha_1\, d\alpha_2$$

$$\times \int e^{-(\alpha_1 + \alpha_2) k^2} d^d k$$

$$= \frac{1}{\Gamma(n_1)\Gamma(n_2)}$$

$$\times \int \exp\left[-\frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2}\right] (\alpha_1 + \alpha_2)^{-d/2} \alpha_1^{n_1 - 1} \alpha_2^{n_2 - 1} d\alpha_1\, d\alpha_2$$

1-loop massless propagator diagram

Substitution $\alpha_1 = \eta x$, $\alpha_2 = \eta(1-x)$

$$G(n_1, n_2) = \frac{1}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 x^{n_1-1}(1-x)^{n_2-1} dx$$
$$\times \int_0^\infty e^{-\eta x(1-x)} \eta^{-d/2+n_1+n_2-1} d\eta$$
$$= \frac{\Gamma(-d/2+n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 x^{d/2-n_2-1}(1-x)^{d/2-n_1-1} dx$$

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$$G(n_1, n_2) = \frac{\Gamma(-d/2+n_1+n_2)\Gamma(d/2-n_1)\Gamma(d/2-n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(d-n_1-n_2)}$$

1-loop massless propagator diagram

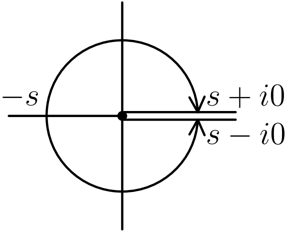
For integer $n_{1,2}$, proportional to

$$G_1 = -\frac{2g_1}{(d-3)(d-4)}$$

$$g_1 = \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

UV divergence if $n_1 + n_2 \leq 2$
IR divergence if $n_1 \geq 2$ or $n_2 \geq 2$

Analytical properties



$$I(s \pm i0) = G_1 s^{-\varepsilon} e^{\pm i\pi\varepsilon}$$

$$I(s+i0) - I(s-i0)$$

$$= G_1 s^{-\varepsilon} 2\varepsilon \sin(\pi\varepsilon) \to 2\pi i$$

$$I(p^2) = -\frac{i}{\pi^{d/2}} \int \frac{d^d k}{(-k^2 - i0)(-(k+p)^2 - i0)} = G_1 (-p^2)^{-\varepsilon}$$

Massless QCD

$$L = \sum_i \bar{q}_{0i} i \not{D} q_{0i} - \frac{1}{4} G^a_{0\mu\nu} G_0^{a\mu\nu}$$

$$D_\mu q_0 = \left(\partial_\mu - i g_0 A_{0\mu}\right) q_0 \qquad A_{0\mu} = A^a_{0\mu} t^a$$

$$[D_\mu, D_\nu] q_0 = -i g_0 G_{0\mu\nu} q_0 \qquad G_{0\mu\nu} = G^a_{0\mu\nu} t^a$$

$$G^a_{0\mu\nu} = \partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu} + g_0 f^{abc} A^b_{0\mu} A^c_{0\nu}$$

Massless QCD

$$L = \sum_i \bar{q}_{0i} i \not{D} q_{0i} - \frac{1}{4} G^a_{0\mu\nu} G^{a\mu\nu}_0 - \frac{1}{2a_0} \left(\partial_\mu A^{a\mu}_0\right)^2 + (\partial^\mu \bar{c}^a_0)(D_\mu c^a_0)$$

$$D_\mu q_0 = \left(\partial_\mu - i g_0 A_{0\mu}\right) q_0 \qquad A_{0\mu} = A^a_{0\mu} t^a$$

$$[D_\mu, D_\nu] q_0 = -i g_0 G_{0\mu\nu} q_0 \qquad G_{0\mu\nu} = G^a_{0\mu\nu} t^a$$

$$G^a_{0\mu\nu} = \partial_\mu A^a_{0\nu} - \partial_\nu A^a_{0\mu} + g_0 f^{abc} A^b_{0\mu} A^c_{0\nu}$$

$$D_\mu c^a_0 = (\partial_\mu \delta^{ab} - i g_0 A^{ab}_{0\mu}) c^b_0 \qquad A^{ab}_{0\mu} = A^c_{0\mu} (t^c)^{ab}$$

$$[t^a, t^b] = i f^{abc} t^c \qquad (t^c)^{ab} = i f^{acb}$$

Renormalization

Renormalized quantities

$$q_0 = Z_q^{1/2} q \qquad A_0 = Z_A^{1/2} A \qquad a_0 = Z_A a \qquad g_0 = Z_\alpha^{1/2} g$$

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Minimal renormalization constants

$$Z_i(\alpha_s) = 1 + \frac{z_1}{\varepsilon} \frac{\alpha_s}{4\pi} + \left(\frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon} \right) \left(\frac{\alpha_s}{4\pi} \right)^2 + \cdots$$

Renormalization

Renormalized quantities

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Dimensionalities: $[L] = d$, $[A_0] = 1 - \varepsilon$, $[q_0] = 3/2 - \varepsilon$, $[g_0] = \varepsilon$. Exactly dimensionless

$$\frac{\alpha_s(\mu)}{4\pi} = \mu^{-2\varepsilon}\frac{g^2}{(4\pi)^{d/2}}e^{-\gamma\varepsilon} \qquad \frac{g_0^2}{(4\pi)^{d/2}} = \mu^{2\varepsilon}\frac{\alpha_s(\mu)}{4\pi}Z_\alpha(\alpha(\mu))e^{\gamma\varepsilon}$$

Gluon propagator

$$-iD_{\mu\nu}(p) = -iD^0_{\mu\nu}(p) + (-i)D^0_{\mu\alpha}(p)i\Pi^{\alpha\beta}(p)(-i)D^0_{\beta\nu}(p)$$
$$+ (-i)D^0_{\mu\alpha}(p)i\Pi^{\alpha\beta}(p)(-i)D^0_{\beta\gamma}(p)i\Pi^{\gamma\delta}(p)(-i)D^0_{\gamma\nu}(p) + \cdots$$

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$$D_{\mu\nu}(p) = D^0_{\mu\nu}(p) + D^0_{\mu\alpha}(p)\Pi^{\alpha\beta}(p)D_{\beta\nu}(p)$$

Gluon propagator

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$$D_{\mu\nu}(p) = D^0_{\mu\nu}(p) + D^0_{\mu\alpha}(p)\Pi^{\alpha\beta}(p)D_{\beta\nu}(p)$$

$$\left.\begin{aligned} A_{\mu\nu} &= A_\perp\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right] + A_{||}\frac{p_\mu p_\nu}{p^2} \\ A^{-1}_{\mu\nu} &= A^{-1}_\perp\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right] + A^{-1}_{||}\frac{p_\mu p_\nu}{p^2} \end{aligned}\right\} \quad A^{-1}_{\mu\lambda}A^{\lambda\nu} = \delta^\nu_\mu$$

Gluon propagator

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$$D^{-1}_{\mu\nu}(p) = (D^0)^{-1}_{\mu\nu}(p) - \Pi_{\mu\nu}(p)$$

Gluon propagator

Slavnov–Taylor identity $\Pi_{\mu\nu}(p)p^{\nu} = 0$

$$\Pi_{\mu\nu}(p) = (p^2 g_{\mu\nu} - p_\mu p_\nu)\Pi(p^2)$$

Gluon propagator

Slavnov–Taylor identity $\Pi_{\mu\nu}(p)p^{\nu} = 0$

$$\Pi_{\mu\nu}(p) = (p^2 g_{\mu\nu} - p_\mu p_\nu)\Pi(p^2)$$

$$D_{\mu\nu}(p) = \frac{1}{p^2(1-\Pi(p^2))}\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right] + a_0 \frac{p_\mu p_\nu}{(p^2)^2}$$

Gluon propagator

Slavnov–Taylor identity $\Pi_{\mu\nu}(p)p^\nu = 0$

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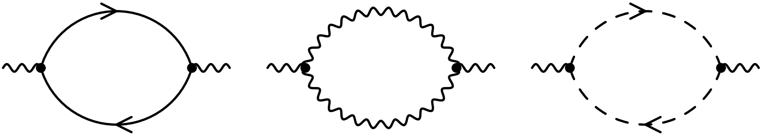
Renormalized propagator $D_{\mu\nu}(p) = Z_A(\alpha(\mu)) D^r_{\mu\nu}(p;\mu)$

$$D^r_{\mu\nu}(p;\mu) = D^r_\perp(p^2;\mu)\left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right] + a(\mu)\frac{p_\mu p_\nu}{(p^2)^2}$$

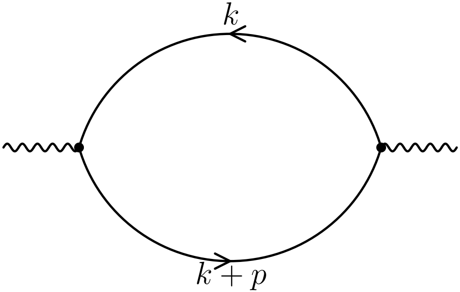
Slavnov–Taylor identities

Gluon self-energy



$$i\delta^{ab}\Pi_{\mu\nu}(p) \qquad \Pi_{\mu\nu}(p) = (p^2 g_{\mu\nu} - p_\mu p_\nu)\Pi(p^2)$$

Quark contribution

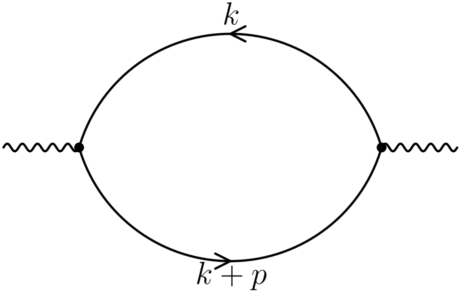


Fermion loop -1

$$i(p^2 g_{\mu\nu} - p_\mu p_\nu)\Pi(p^2) =$$

$$- T_F n_f \int \frac{d^d k}{(2\pi)^d} \operatorname{Tr} i g_0 \gamma_\mu i \frac{\not{k} + \not{p}}{(k+p)^2} i g_0 \gamma_\nu \frac{\not{k}}{k^2}$$

Quark contribution



Fermion loop -1

$$i(p^2 g_{\mu\nu} - p_\mu p_\nu)\Pi(p^2) =$$

$$-T_F n_f \int \frac{d^d k}{(2\pi)^d} \operatorname{Tr} i g_0 \gamma_\mu i \frac{\not{k} + \not{p}}{(k+p)^2} i g_0 \gamma_\nu \frac{\not{k}}{k^2}$$

In d dimensions $\delta^\mu_\mu = d$

$$\Pi(p^2) = \frac{-i T_F n_f g_0^2}{(d-1)(-p^2)} \int \frac{d^d k}{(2\pi)^d} \frac{\operatorname{Tr} \gamma_\mu (\not{k} + \not{p}) \gamma^\mu \not{k}}{\left[-(k+p)^2\right] \left(-k^2\right)}$$

γ matrices in d dimensions

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

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γ matrices in d dimensions

$$
\begin{aligned}
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&\gamma_\mu\gamma^\mu = d \\
&\gamma_\mu \not{a} \gamma^\mu = \gamma_\mu(-\gamma^\mu \not{a} + 2a^\mu) = -(d-2)\not{a} \\
&\gamma_\mu \not{a}\not{b} \gamma^\mu = \gamma_\mu \not{a}(-\gamma^\mu \not{b} + 2b^\mu) = (d-2)\not{a}\not{b} + 2\not{b}\not{a} \\
&\qquad = 4a\cdot b + (d-4)\not{a}\not{b}
\end{aligned}
$$

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&\gamma_\mu \not{a}\not{b}\not{c}\gamma^\mu = \gamma_\mu\not{a}\not{b}(-\gamma^\mu\not{c} + 2c^\mu) = -4a\cdot b - (d-4)\not{a}\not{b}\not{c} + 2\not{c}\not{a}\not{b} \\
&\qquad = -2\not{c}\not{b}\not{a} - (d-4)\not{a}\not{b}\not{c}
\end{aligned}
$$

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Convention: $\mathrm{Tr}\, 1 = 4$

Gluon self-energy – quark contribution

$$\Pi(p^2) = \frac{d-2}{d-1} \frac{iT_F n_f g_0^2}{-p^2} \int \frac{d^d k}{(2\pi)^d} \frac{4(k+p)\cdot k}{\left[-(k+p)^2\right]\left(-k^2\right)}$$

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Set $-p^2 = 1$

$$D_1 = -(k+p)^2 \qquad D_2 = -k^2$$

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Multiplication table

$$p^2 = -1 \qquad k^2 = -D_2 \qquad p\cdot k = \frac{1}{2}\left(1 + D_2 - D_1\right)$$

Gluon self-energy – quark contribution

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Multiplication table

$$p^2 = -1 \qquad k^2 = -D_2 \qquad p\cdot k = \frac{1}{2}\left(1 + D_2 - D_1\right)$$

$$\Pi(p^2) = 2\frac{d-2}{d-1}iT_F n_f e_0^2\int\frac{d^d k}{(2\pi)^d}\frac{-2D_2 + 1 + D_2 - D_1}{D_1 D_2}$$

Gluon self-energy – quark contribution

Restoring $-p^2$ by dimensionality

$$\Pi(p^2) = -T_F n_f \frac{g_0^2 (-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} 2 \frac{d-2}{d-1} G_1$$

Gluon self-energy – quark contribution

Restoring $-p^2$ by dimensionality

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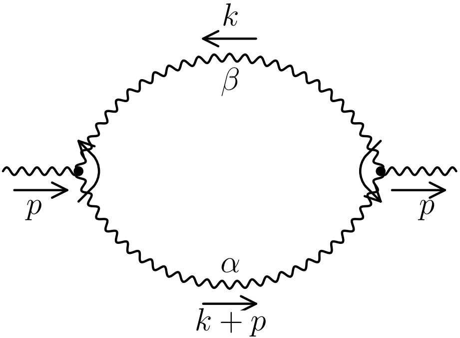
Recalling G_1

$$\Pi(p^2) = T_F n_f \frac{g_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} 4 \frac{d-2}{(d-1)(d-3)(d-4)} g_1$$

Gluon contribution



Symmetry factor $\frac{1}{2}$. Colour factor C_A. Feynman gauge $a_0 = 1$

$$\Pi_1{}^{\mu}_{\mu} = -i\frac{1}{2}C_A g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{N}{k^2(k+p)^2}$$

$$N = V_{\mu\alpha\beta}(p, -k-p, k) V^{\mu\beta\alpha}(-p, -k, k+p)$$

Gluon contribution

$$V_{\mu\alpha\beta}(p,-k-p,k) = (2k+p)_\mu g_{\alpha\beta} - (k-p)_\alpha g_{\beta\mu} - (k+2p)_\beta g_{\mu\alpha}$$

$V^{\mu\beta\alpha}(-p,-k,k+p)$ — the same

$$N = d\left[(2k+p)^2 + (k-p)^2 + (k+2p)^2\right]$$
$$- 2(2k+p)\cdot(k-p) - 2(2k+p)\cdot(k+2p) + 2(k+2p)\cdot(k-p)$$

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$$p^2 = -1 \qquad k^2 = -D_2 \Rightarrow 0 \qquad p\cdot k = \frac{1}{2}(1 - D_1 + D_2) \Rightarrow \frac{1}{2}$$

$$N \Rightarrow -3(d-1)$$

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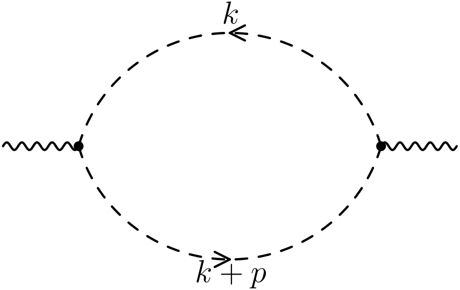
$$N \Rightarrow -3(d-1)$$

$$\Pi_1{}^\mu_\mu = -\frac{3}{2} C_A \frac{g_0^2(-p^2)^{1-\varepsilon}}{(4\pi)^{d/2}} G_1 (d-1)$$

Ghost contribution



Fermion loop gives -1. Colour factor C_A

$$\Pi_{2\mu}^{\ \mu} = iC_A g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{k \cdot (k+p)}{k^2 (k+p)^2} = -\frac{1}{2} C_A \frac{g_0^2 (-p^2)^{1-\varepsilon}}{(4\pi)^{d/2}} G_1$$

Result

$$\Pi_g(p^2) = -\frac{\Pi_1{}_\mu^\mu + \Pi_2{}_\mu^\mu}{(d-1)(-p^2)} = C_A \frac{g_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} G_1 \frac{3d-2}{2(d-1)}$$

Result

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In an arbitrary covariant gauge $\xi = 1 - a_0$

$$\Pi_g(p^2) = C_A \frac{g_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} \frac{G_1}{2(d-1)} \times \left[3d - 2 + (d-1)(2d-7)\xi - \frac{1}{4}(d-1)(d-4)\xi^2 \right]$$

Gluon field renormalization

$$p^2 D_\perp(p^2) = 1 + \frac{\alpha_s(\mu)}{4\pi\varepsilon} e^{-L\varepsilon} \left[-\frac{1}{2}\left(a - \frac{13}{3}\right) C_A - \frac{4}{3} T_F n_f \right.$$
$$\left. + \left(\frac{9a^2 + 18a + 97}{36} C_A - \frac{20}{9} T_F n_f \right) \varepsilon \right]$$

$$L = \log \frac{-p^2}{\mu^2}$$

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$$L = \log \frac{-p^2}{\mu^2}$$

$$Z_A = 1 - \frac{\alpha_s}{4\pi\varepsilon} \left[\frac{1}{2}\left(a - \frac{13}{3}\right) C_A + \frac{4}{3} T_F n_f \right]$$
$$\gamma_A = \left[\left(a - \frac{13}{3}\right) C_A + \frac{8}{3} T_F n_f \right] \frac{\alpha_s}{4\pi} + \cdots$$

Photon field renormalization

Transverse propagator

$$p^2 D_\perp(p^2) = \frac{1}{1-\Pi(p^2)} = 1 + \frac{e_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} 4 \frac{d-2}{(d-1)(d-3)(d-4)} g_1$$

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Re-expressing via $\alpha(\mu)$

$$p^2 D_\perp(p^2) = 1 + \frac{\alpha(\mu)}{4\pi} e^{-L\varepsilon} e^{\gamma\varepsilon} g_1 \, 4 \frac{d-2}{(d-1)(d-3)(d-4)}$$

$$L = \log \frac{-p^2}{\mu^2}$$

Photon field renormalization

For g_1 at $\varepsilon \to 0$, using

$$\Gamma(1+\varepsilon) = \exp\left[-\gamma\varepsilon + \sum_{n=2}^{\infty} \frac{(-1)^n \zeta_n}{n} \varepsilon^n\right]$$

$$\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n} \qquad \zeta_2 = \frac{\pi^2}{6} \qquad \zeta_3 \approx 1.202 \qquad \zeta_4 = \frac{\pi^4}{90} \qquad \cdots$$

we have $e^{\gamma\varepsilon} g_1 = 1 + \mathcal{O}(\varepsilon^2)$

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$$p^2 D_\perp(p^2) = 1 - \frac{4}{3}\frac{\alpha(\mu)}{4\pi\varepsilon}\left[1 - \left(L - \frac{5}{3}\right)\varepsilon + \cdots\right]$$

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$$p^2 D_\perp(p^2) = 1 - \frac{4}{3}\frac{\alpha(\mu)}{4\pi\varepsilon}\left[1 - \left(L - \frac{5}{3}\right)\varepsilon + \cdots\right]$$

This should be $Z_A(\alpha(\mu)) p^2 D_\perp^r(p^2;\mu)$:

$$Z_A(\alpha) = 1 - \frac{4}{3}\frac{\alpha}{4\pi\varepsilon}$$

$$p^2 D_\perp^r(p^2;\mu) = 1 + \frac{4}{3}\frac{\alpha(\mu)}{4\pi}\left(L - \frac{5}{3}\right)$$

RG equation

$D_\perp(p^2) = Z_A(\alpha(\mu)) D^r_\perp(p^2;\mu)$ does not depend on μ:

$$\frac{\partial D^r_\perp(p^2;\mu)}{\partial \log \mu} + \gamma_A(\alpha(\mu)) D^r_\perp(p^2;\mu) = 0$$

$$\gamma_A(\alpha(\mu)) = \frac{d \log Z_A(\alpha(\mu))}{d \log \mu}$$

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For a minimal renormalization constant, using

$$\frac{d \log \alpha(\mu)}{d \log \mu} = -2\varepsilon + \cdots$$

we obtain

$$\gamma(\alpha) = \gamma_0 \frac{\alpha}{4\pi} + \cdots = -2z_1 \frac{\alpha}{4\pi} + \cdots$$

$$Z(\alpha) = 1 - \frac{\gamma_0}{2} \frac{\alpha}{4\pi\varepsilon} + \cdots$$

Anomalous dimension

$$\gamma_A(\alpha) = \frac{8}{3}\frac{\alpha}{4\pi}$$

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RG equation

$$\frac{\partial p^2 D^r_\perp}{\partial L} = \frac{\gamma_A}{2} p^2 D^r_\perp$$

with the initial condition

$$p^2 D^r_\perp(p^2; \mu^2 = -p^2) = 1 - \frac{20}{9}\frac{\alpha(\mu)}{4\pi}$$

is enough to reconstruct the result

Anomalous dimension

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with the initial condition

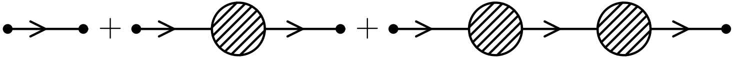
$$p^2 D^r_\perp(p^2; \mu^2 = -p^2) = 1 - \frac{20}{9}\frac{\alpha(\mu)}{4\pi}$$

is enough to reconstruct the result

$a_0 = Z_A(\alpha(\mu))a(\mu)$ does not depend on μ:

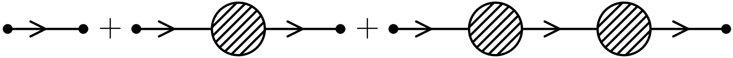
$$\frac{da(\mu)}{d\log\mu} + \gamma_A(\alpha(\mu))a(\mu) = 0$$

Quark propagator



$$iS(p) = iS_0(p) + iS_0(p)(-i)\Sigma(p)iS_0(p) + iS_0(p)(-i)\Sigma(p)iS_0(p)(-i)\Sigma(p)iS_0(p) + \cdots$$

Quark propagator



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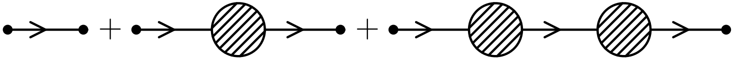
$$S(p) = S_0(p) + S_0(p)\Sigma(p)S(p)$$

Quark propagator



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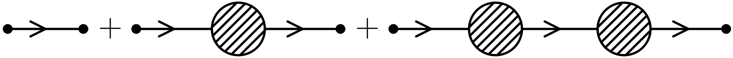
$$S(p) = S_0(p) + S_0(p)\Sigma(p)S(p)$$

$$S(p) = \frac{1}{S_0^{-1}(p) - \Sigma(p)}$$

Quark propagator



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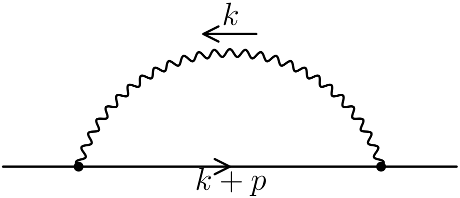
$$S(p) = S_0(p) + S_0(p)\Sigma(p)S(p)$$

$$S(p) = \frac{1}{S_0^{-1}(p) - \Sigma(p)}$$

Massless case: $\Sigma(p) = \not{p}\Sigma_V(p^2)$ (helicity conservation)

$$S(p) = \frac{1}{1 - \Sigma_V(p^2)}\frac{1}{\not{p}}$$

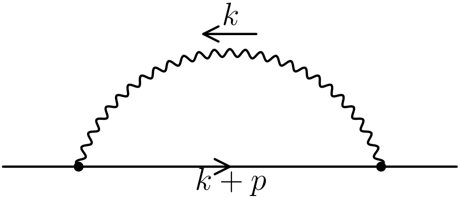
Quark self-energy



$$-i\not{p}\Sigma_V(p^2) = C_F \int \frac{d^dk}{(2\pi)^d} ig_0\gamma^\mu i \frac{\not{k}+\not{p}}{(k+p)^2} ig_0\gamma^\nu \frac{-i}{k^2}\left(g_{\mu\nu} - \xi\frac{k_\mu k_\nu}{k^2}\right)$$

where $\xi = 1 - a_0$.

Quark self-energy



$$-i\not{p}\Sigma_V(p^2) = C_F \int \frac{d^dk}{(2\pi)^d} ig_0\gamma^\mu i \frac{\not{k}+\not{p}}{(k+p)^2} ig_0\gamma^\nu \frac{-i}{k^2}\left(g_{\mu\nu} - \xi\frac{k_\mu k_\nu}{k^2}\right)$$

where $\xi = 1 - a_0$. Taking $\frac{1}{4}\operatorname{Tr}\not{p}$

$$\Sigma_V(p^2) = \frac{iC_F g_0^2}{-p^2}\int \frac{d^dk}{(2\pi)^d}\frac{N}{D_1 D_2}$$

$$N = \frac{1}{4}\operatorname{Tr}\not{p}\gamma^\mu(\not{k}+\not{p})\gamma^\nu\left(g_{\mu\nu} + \xi\frac{k_\mu k_\nu}{D_2}\right)$$

Quark self-energy

Using multiplication table

$$N = \frac{1}{4}\,\mathrm{Tr}\,\not{p}\gamma_\mu(\not{k}+\not{p})\gamma^\mu + \frac{\xi}{D_2}\,\frac{1}{4}\,\mathrm{Tr}\,\not{p}\not{k}(\not{k}+\not{p})\not{k}$$

$$= -(d-2)(p^2+p\cdot k) + \frac{\xi}{D_2}\left[k^2 p\cdot k + 2(p\cdot k)^2 - p^2 k^2\right]$$

$$= \frac{1}{2}\left[d-2+\xi\left(\frac{1}{D_2}-1\right)\right]$$

Quark self-energy

Using multiplication table

$$N = \frac{1}{4} \operatorname{Tr} \not{p} \gamma_\mu (\not{k} + \not{p}) \gamma^\mu + \frac{\xi}{D_2} \frac{1}{4} \operatorname{Tr} \not{p} \not{k} (\not{k} + \not{p}) \not{k}$$

$$= -(d-2)(p^2 + p \cdot k) + \frac{\xi}{D_2} \left[k^2 p \cdot k + 2(p \cdot k)^2 - p^2 k^2 \right]$$

$$= \frac{1}{2} \left[d - 2 + \xi \left(\frac{1}{D_2} - 1 \right) \right]$$

$$\Sigma_V(p^2) = -C_F \frac{g_0^2 (-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} \frac{1}{2} \left[(d - 2 - \xi) G(1,1) + \xi G(1,2) \right]$$

Quark self-energy

$$\frac{G(n_1, n_2 + 1)}{G(n_1, n_2)} = -\frac{(d - 2n_1 - 2n_2)(d - n_1 - n_2 - 1)}{n_2(d - 2n_2 - 2)}$$

Quark self-energy

$$\frac{G(n_1, n_2 + 1)}{G(n_1, n_2)} = -\frac{(d - 2n_1 - 2n_2)(d - n_1 - n_2 - 1)}{n_2(d - 2n_2 - 2)}$$

$$\frac{G(1, 2)}{G(1, 1)} = -(d - 3)$$

Quark self-energy

$$\frac{G(n_1, n_2 + 1)}{G(n_1, n_2)} = -\frac{(d - 2n_1 - 2n_2)(d - n_1 - n_2 - 1)}{n_2(d - 2n_2 - 2)}$$

$$\frac{G(1,2)}{G(1,1)} = -(d-3)$$

$$\Sigma_V(p^2) = -C_F \frac{g_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} \frac{d-2}{2} a_0 G_1$$

Quark self-energy

$$\frac{G(n_1, n_2 + 1)}{G(n_1, n_2)} = -\frac{(d - 2n_1 - 2n_2)(d - n_1 - n_2 - 1)}{n_2(d - 2n_2 - 2)}$$

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$$\Sigma_V(p^2) = C_F \frac{g_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} \frac{d-2}{(d-3)(d-4)} a_0 g_1$$

Quark field renormalization

$$\not{p}S(p) = \frac{1}{1 - \Sigma_V(p^2)}$$

expressed via the renormalized quantities

$$\not{p}S(p) = 1 + C_F \frac{\alpha_s(\mu)}{4\pi} e^{-L\varepsilon} e^{\gamma\varepsilon} g_1 \, a(\mu) \frac{d-2}{(d-3)(d-4)}$$

$$= 1 - \frac{\alpha(\mu)}{4\pi\varepsilon} a(\mu) e^{-L\varepsilon} (1 + \varepsilon + \cdots)$$

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should be $Z_q(\alpha(\mu), a(\mu)) \not{p} S_r(p;\mu)$:

$$Z_q(\alpha, a) = 1 - C_F a \frac{\alpha_s}{4\pi\varepsilon}$$

$$\not{p} S_r(p) = 1 + C_F a(\mu)(L-1) \frac{\alpha_s(\mu)}{4\pi}$$

Quark field renormalization

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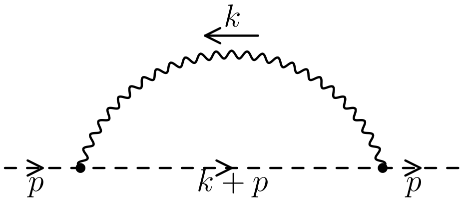
$$\not{p}S_r(p) = 1 + C_F a(\mu)(L-1)\frac{\alpha_s(\mu)}{4\pi}$$

$$\gamma_q(\alpha_s, a) = 2a\frac{\alpha_s}{4\pi}$$

Ghost self-energy



$$\Sigma(p^2) = -iC_A g_0^2 \int \frac{d^dk}{(2\pi)^d} \frac{p^\mu (k+p)^\nu}{k^2 (k+p)^2} \left(g_{\mu\nu} - \xi \frac{k_\mu k_\nu}{k^2} \right)$$

$$= C_A \frac{g_0^2 (-p^2)^{1-\varepsilon}}{(4\pi)^{d/2}} \left[-\frac{1}{2} G(1,1) + \frac{\xi}{4} G(1,2) \right]$$

$$= -\frac{1}{4} C_A \frac{g_0^2 (-p^2)^{1-\varepsilon}}{(4\pi)^{d/2}} G_1 \left[d - 1 - (d-3) a_0 \right]$$

Ghost field renormalization

$$G(p) = \frac{1}{p^2 - \Sigma(p^2)} = \frac{1}{p^2}\left[1 + C_A \frac{\alpha_s(\mu)}{4\pi\varepsilon} e^{-L\varepsilon} \frac{3 - a + 4\varepsilon}{4}\right]$$

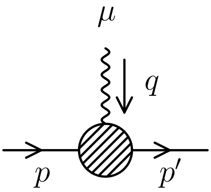
Ghost field renormalization

$$G(p) = \frac{1}{p^2 - \Sigma(p^2)} = \frac{1}{p^2} \left[1 + C_A \frac{\alpha_s(\mu)}{4\pi\varepsilon} e^{-L\varepsilon} \frac{3 - a + 4\varepsilon}{4} \right]$$

$$Z_c = 1 + C_A \frac{3 - a}{4} \frac{\alpha_s}{4\pi\varepsilon}$$

$$\gamma_c = -C_A \frac{3 - a}{2} \frac{\alpha_s}{4\pi} + \cdots$$

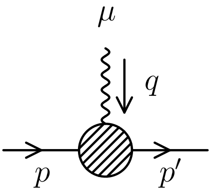
Vertex



$$= ig_0 t^a \Gamma^\mu(p, p')$$

$$\Gamma^\mu(p, p') = \gamma^\mu + \Lambda^\mu(p, p')$$

Vertex



$$= ig_0 t^a \Gamma^\mu(p, p')$$

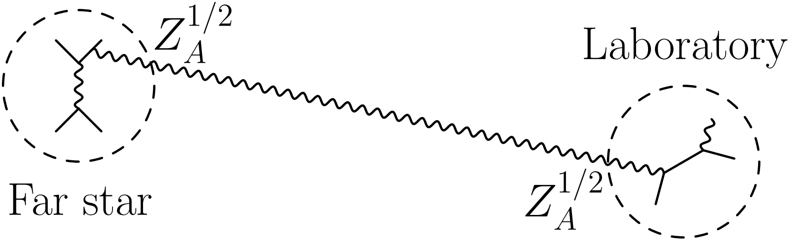
$$\Gamma^\mu(p, p') = \gamma^\mu + \Lambda^\mu(p, p')$$

When expressed via renormalized quantities,

$$\Gamma^\mu = Z_\Gamma \Gamma_r^\mu$$

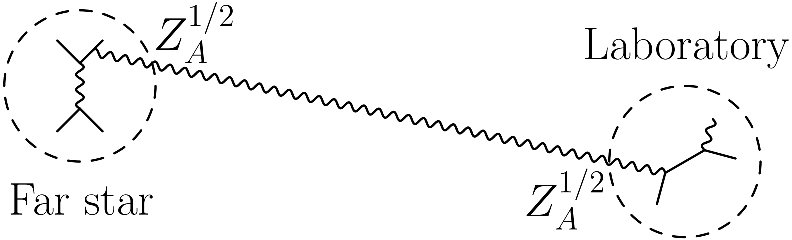
Matrix element

S-matrix element $=$ vertex $\times$ $Z_i^{1/2}$ for each i



Matrix element

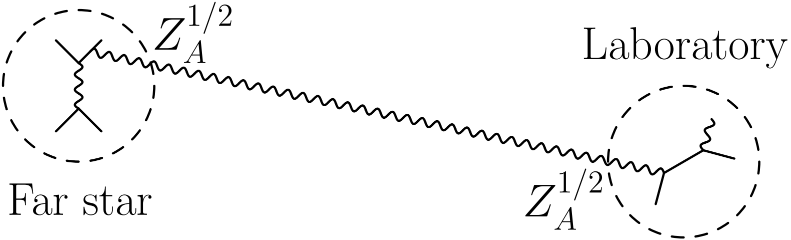
S-matrix element $=$ vertex $\times$ $Z_i^{1/2}$ for each i



The physical matrix element
$e_0 \Gamma Z_\psi Z_A^{1/2} = e \Gamma_r Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2}$ must be finite

Matrix element

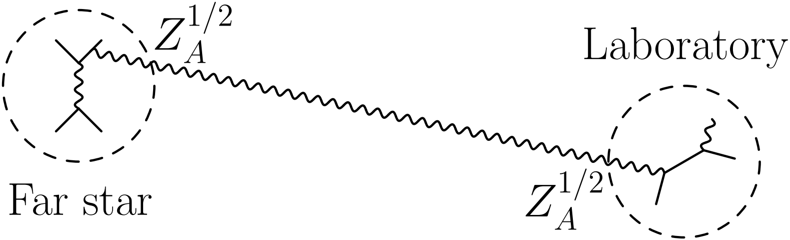
S-matrix element = vertex $\times$ $Z_i^{1/2}$ for each i



The physical matrix element
$e_0 \Gamma Z_\psi Z_A^{1/2} = e \Gamma_r Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2}$ must be finite
$Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2}$ must be finite

Matrix element

S-matrix element $=$ vertex $\times$ $Z_i^{1/2}$ for each i



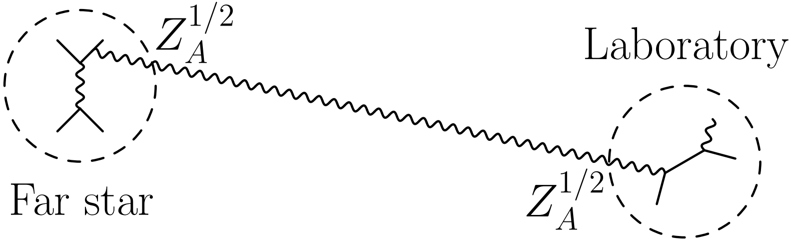
The physical matrix element

$e_0 \Gamma Z_\psi Z_A^{1/2} = e \Gamma_r Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2}$ must be finite

$Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2}$ must be finite $= 1$

Matrix element

S-matrix element $=$ vertex $\times$ $Z_i^{1/2}$ for each i



The physical matrix element
$e_0 \Gamma Z_\psi Z_A^{1/2} = e \Gamma_r Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2}$ must be finite
$Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2}$ must be finite $= 1$

$$Z_\alpha = (Z_\Gamma Z_\psi)^{-2} Z_A^{-1}$$

Quark–gluon vertex

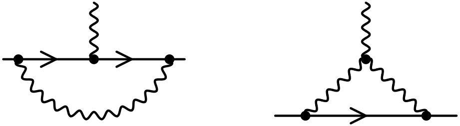
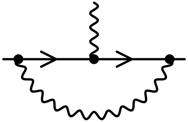


Diagram 1: like in QED, but with colour factor

$$\Lambda_1^\alpha = \left(C_F - \frac{C_A}{2}\right)\frac{\alpha_s}{4\pi\varepsilon}\gamma^\alpha$$

Diagram 1

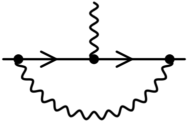
Direct calculation



$$ie_0\Lambda^\alpha = \int \frac{d^d k}{(2\pi)^d} ie_0\gamma^\mu i\frac{\not{k}}{k^2} ie_0\gamma^\alpha i\frac{\not{k}}{k^2} ie_0\gamma^\nu \times \frac{-i}{k^2}\left(g_{\mu\nu} - \xi\frac{k_\mu k_\nu}{k^2}\right)$$

$$\Lambda^\alpha = -ie_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_\mu \not{k}\gamma^\alpha \not{k}\gamma^\mu - \xi k^2\gamma^\alpha}{(k^2)^2}$$

Direct calculation



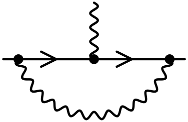
$$ie_0\Lambda^\alpha = \int \frac{d^dk}{(2\pi)^d} ie_0\gamma^\mu i\frac{\not{k}}{k^2} ie_0\gamma^\alpha i\frac{\not{k}}{k^2} ie_0\gamma^\nu \times \frac{-i}{k^2}\left(g_{\mu\nu} - \xi\frac{k_\mu k_\nu}{k^2}\right)$$

$$\Lambda^\alpha = -ie_0^2 \int \frac{d^dk}{(2\pi)^d} \frac{\gamma_\mu\not{k}\gamma^\alpha\not{k}\gamma^\mu - \xi k^2\gamma^\alpha}{(k^2)^2}$$

Averaging $\not{k}\gamma^\alpha\not{k} \to (k^2/d)\gamma_\nu\gamma^\alpha\gamma^\nu$

$$\Lambda^\alpha = -ie_0^2 a_0\gamma^\alpha \int \frac{d^dk}{(2\pi)^d}\frac{1}{(-k^2)^2}$$

Direct calculation



$$ie_0\Lambda^\alpha = \int \frac{d^dk}{(2\pi)^d} ie_0\gamma^\mu i\frac{\not{k}}{k^2} ie_0\gamma^\alpha i\frac{\not{k}}{k^2} ie_0\gamma^\nu \times \frac{-i}{k^2}\left(g_{\mu\nu} - \xi\frac{k_\mu k_\nu}{k^2}\right)$$

$$\Lambda^\alpha = -ie_0^2 \int \frac{d^dk}{(2\pi)^d} \frac{\gamma_\mu \not{k}\gamma^\alpha\not{k}\gamma^\mu - \xi k^2\gamma^\alpha}{(k^2)^2}$$

Averaging $\not{k}\gamma^\alpha\not{k} \to (k^2/d)\gamma_\nu\gamma^\alpha\gamma^\nu$

$$\Lambda^\alpha = -ie_0^2 a_0 \gamma^\alpha \int \frac{d^dk}{(2\pi)^d} \frac{1}{(-k^2)^2}$$

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{(-k^2)^2}\bigg|_{UV} = \frac{i}{8\pi^2}\int_\lambda^\infty k^{-1-2\varepsilon}dk = \frac{i\lambda^{-2\varepsilon}}{(4\pi)^2\varepsilon} = \frac{i}{(4\pi)^2}\frac{1}{\varepsilon}$$

Direct calculation

Any infrared regulator can be used

$$\int \frac{d^dk}{(2\pi)^d} \frac{1}{(-k^2)^2}\bigg|_{UV} = \int \frac{d^dk}{(2\pi)^d} \frac{1}{(m^2 - k^2)^2}$$
$$= \frac{im^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon) = \frac{i}{(4\pi)^2}\frac{1}{\varepsilon}$$

Direct calculation

Any infrared regulator can be used

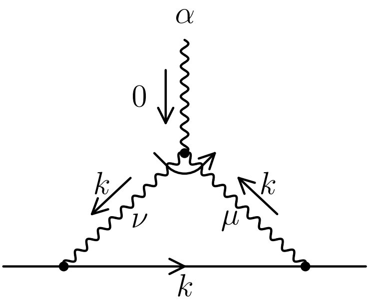
$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(-k^2)^2}\bigg|_{UV} = \int \frac{d^d k}{(2\pi)^d} \frac{1}{(m^2 - k^2)^2}$$
$$= \frac{i m^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) = \frac{i}{(4\pi)^2} \frac{1}{\varepsilon}$$

$$\Gamma^\alpha = \gamma^\alpha \left[1 + a(\mu) \frac{\alpha(\mu)}{4\pi\varepsilon}\right]$$

$$Z_\Gamma = 1 + a \frac{\alpha}{4\pi\varepsilon}$$

agrees with Z_ψ

Diagram 2



Feynman gauge $a_0 = 1$

Colour factor $C_A/2$

$$\Lambda_2^\alpha = i\frac{C_A}{2} g_0^2 \int \frac{d^dk}{(2\pi)^d} \frac{\gamma_\mu \not{k} \gamma_\nu}{(k^2)^3} V^{\alpha\nu\mu}(0, -k, k)$$

$$V^{\alpha\nu\mu}(0, -k, k) = 2k^\alpha g^{\mu\nu} - k^\mu g^{\nu\alpha} - k^\nu g^{\mu\alpha}$$

$$\Lambda_2^\alpha = i\frac{C_A}{2} g_0^2 \int \frac{d^dk}{(2\pi)^d} \frac{2\gamma_\mu \not{k} \gamma^\mu k^\alpha - 2k^2\gamma^\alpha}{(k^2)^3}$$

Result

Averaging $\not{k} k^\alpha \to (k^2/d)\gamma^\alpha$

$$\Lambda_2^\alpha = \frac{3}{2} C_A \frac{\alpha_s}{4\pi\varepsilon} \gamma^\alpha$$

Result

Averaging $\not{k} k^\alpha \to (k^2/d)\gamma^\alpha$

$$\Lambda_2^\alpha = \frac{3}{2} C_A \frac{\alpha_s}{4\pi\varepsilon} \gamma^\alpha$$

Arbitrary covariant gauge

$$\Lambda_2^\alpha = \frac{3}{4}(1+a) C_A \frac{\alpha_s}{4\pi\varepsilon} \gamma^\alpha$$

Result

Averaging $\not{k} k^\alpha \to (k^2/d)\gamma^\alpha$

$$\Lambda_2^\alpha = \frac{3}{2} C_A \frac{\alpha_s}{4\pi\varepsilon} \gamma^\alpha$$

Arbitrary covariant gauge

$$\Lambda_2^\alpha = \frac{3}{4}(1+a) C_A \frac{\alpha_s}{4\pi\varepsilon} \gamma^\alpha$$

$$\Lambda^\alpha = \left(C_F a + C_A \frac{a+3}{4} \right) \frac{\alpha_s}{4\pi\varepsilon} \gamma^\alpha$$

Result

Averaging $\not{k} k^\alpha \to (k^2/d)\gamma^\alpha$

$$\Lambda_2^\alpha = \frac{3}{2} C_A \frac{\alpha_s}{4\pi\varepsilon}\gamma^\alpha$$

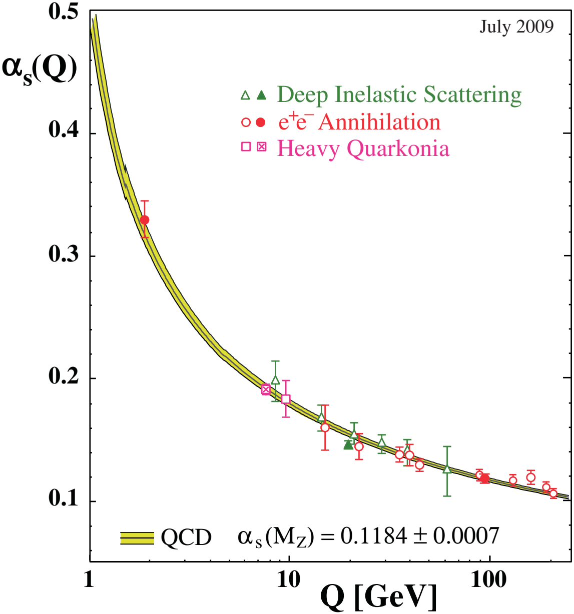
Arbitrary covariant gauge

$$\Lambda_2^\alpha = \frac{3}{4}(1+a) C_A \frac{\alpha_s}{4\pi\varepsilon}\gamma^\alpha$$

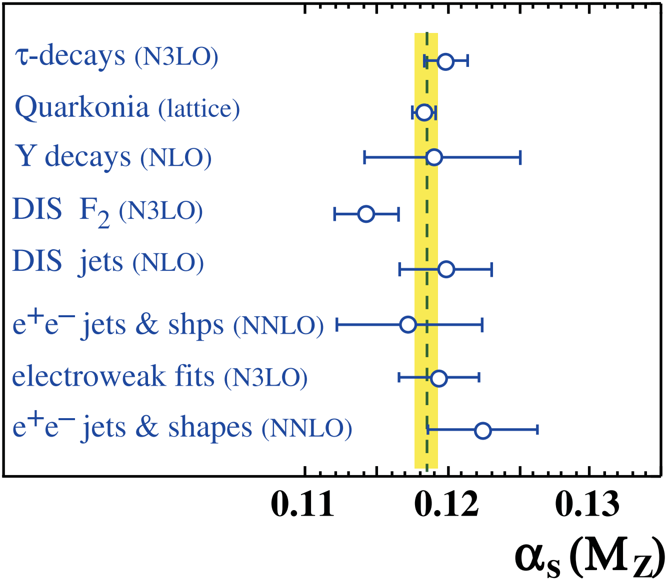
$$\Lambda^\alpha = \left(C_F a + C_A \frac{a+3}{4} \right) \frac{\alpha_s}{4\pi\varepsilon}\gamma^\alpha$$

$$Z_\Gamma = 1 + \left(C_F a + C_A \frac{a+3}{4} \right) \frac{\alpha_s}{4\pi\varepsilon}$$

$\alpha_s(Q)$

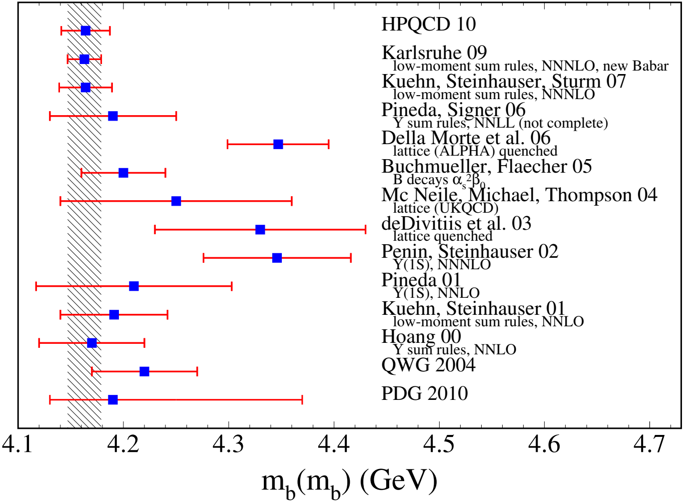

0.5
0.4
0.3
0.2
0.1
$\alpha_s(Q)$
July 2009
Deep Inelastic Scattering
e^+e^- Annihilation
Heavy Quarkonia
QCD $\alpha_s(M_Z) = 0.1184 \pm 0.0007$
1
10
100
Q [GeV]

$\alpha_s(m_Z)$



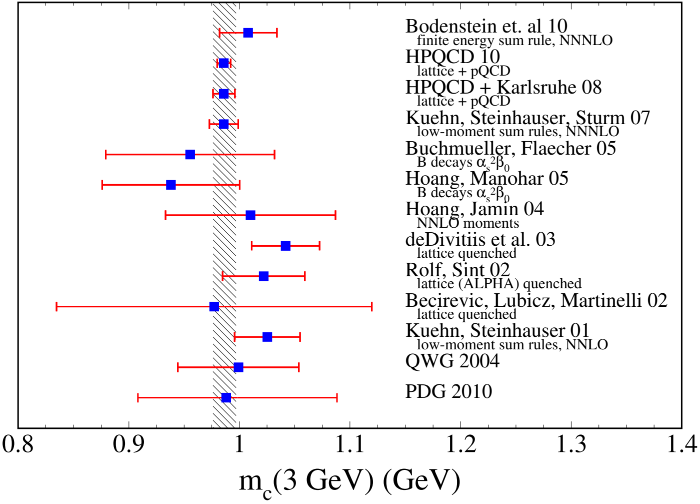
τ-decays (N3LO)
Quarkonia (lattice)
Υ decays (NLO)
DIS F2 (N3LO)
DIS jets (NLO)
e+e− jets & shps (NNLO)
electroweak fits (N3LO)
e+e− jets & shapes (NNLO)
0.11
0.12
0.13
αs (MZ)

$m_b(m_b)$



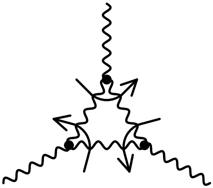
HPQCD 10
Karlsruhe 09
low-moment sum rules, NNNLO, new Babar
Kuehn, Steinhauser, Sturm 07
low-moment sum rules, NNNLO
Pineda, Signer 06
Υ sum rules, NNLL (not complete)
Della Morte et al. 06
lattice (ALPHA) quenched
Buchmueller, Flaecher 05
B decays $\alpha_s^2\beta_0$
Mc Neile, Michael, Thompson 04
lattice (UKQCD)
deDivitiis et al. 03
lattice quenched
Penin, Steinhauser 02
Υ(1S), NNNLO
Pineda 01
Υ(1S), NNLO
Kuehn, Steinhauser 01
low-moment sum rules, NNLO
Hoang 00
Υ sum rules, NNLO
QWG 2004
PDG 2010
4.1
4.2
4.3
4.4
4.5
4.6
4.7
$m_b(m_b)$ (GeV)

$m_c(3\text{ GeV})$

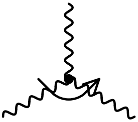


Bodenstein et. al 10
finite energy sum rule, NNNLO
HPQCD 10
lattice + pQCD
HPQCD + Karlsruhe 08
lattice + pQCD
Kuehn, Steinhauser, Sturm 07
low-moment sum rules, NNNLO
Buchmueller, Flaecher 05
B decays $\alpha_s^2\beta_0$
Hoang, Manohar 05
B decays $\alpha_s^2\beta_0$
Hoang, Jamin 04
NNLO moments
deDivitiis et al. 03
lattice quenched
Rolf, Sint 02
lattice (ALPHA) quenched
Becirevic, Lubicz, Martinelli 02
lattice quenched
Kuehn, Steinhauser 01
low-moment sum rules, NNLO
QWG 2004
PDG 2010
0.8
0.9
1
1.1
1.2
1.3
1.4
$m_c(3\text{ GeV})$ (GeV)

Example 3

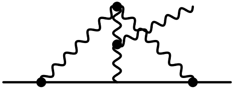


$= \frac{C_A}{2}$



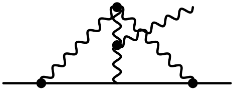
$$if^{adf} if^{bed} if^{cfe} = \frac{C_A}{2} if^{abc}$$

Example 4

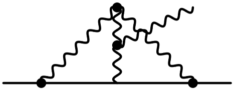


$= 0$

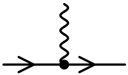
Example 4



$= 0$



$= c$



Example 4