

QCD

Andrey Grozin

Budker Institute of Nuclear Physics
A.G.Grozin@inp.nsk.su

Quark field

q^i colour $i \in [1, N_c]$

Quark field

q^i colour $i \in [1, N_c]$

$$q^i \rightarrow U^i_j q^j \quad \text{or} \quad q \rightarrow Uq$$
$$U^\dagger U = 1 \quad \det U = 1$$

Fundamental representation of $SU(N_c)$

Quark field

q^i colour $i \in [1, N_c]$

$$q^i \rightarrow U^i_j q^j \quad \text{or} \quad q \rightarrow Uq$$
$$U^+ U = 1 \quad \det U = 1$$

Fundamental representation of $SU(N_c)$

$$\bar{q}_i = (q^i)^\dagger \gamma^0$$

$$\bar{q}_i \rightarrow U_i^j \bar{q}_j \quad \text{or} \quad \bar{q} \rightarrow \bar{q} U^+ \quad \text{where} \quad U_i^j = (U^i_j)^*$$

Conjugated fundamental representation

Invariant tensors

Meson

$$\bar{q}q' \rightarrow \bar{q}U^+Uq' = \bar{q}q'$$

$$\delta_j^i \rightarrow \delta_{j'}^{i'} U^{i'} U_j^{j'} = U^i_k U_j^k = \delta_j^i$$

Invariant tensors

Meson

$$\bar{q}q' \rightarrow \bar{q}U^+Uq' = \bar{q}q'$$

$$\delta_j^i \rightarrow \delta_{j'}^{i'}U^i{}_{i'}U_j{}^{j'} = U^i{}_kU_j{}^k = \delta_j^i$$

Baryon ($N_c = 3$)

$$\begin{aligned}\varepsilon_{ijk}q_1^iq_2^jq_3^k &\rightarrow \varepsilon_{ijk}U^i{}_{i'}U^j{}_{j'}U^k{}_{k'}q_1^{i'}q_2^{j'}q_3^{k'} = \det U \cdot \varepsilon_{i'j'k'}q_1^{i'}q_2^{j'}q_3^{k'} \\ &= \varepsilon_{ijk}q_1^iq_2^jq_3^k\end{aligned}$$

$$\varepsilon_{ijk} \rightarrow \varepsilon_{i'j'k'}U^i{}_{i'}U^j{}_{j'}U^k{}_{k'} = \det U^+ \cdot \varepsilon_{ijk} = \varepsilon_{ijk}$$

Invariant tensors

Meson

$$\bar{q}q' \rightarrow \bar{q}U^+Uq' = \bar{q}q'$$

$$\delta_j^i \rightarrow \delta_{j'}^{i'} U^i{}_{i'} U_j{}^{j'} = U^i{}_k U_j{}^k = \delta_j^i$$

Baryon ($N_c = 3$)

$$\begin{aligned} \varepsilon_{ijk} q_1^i q_2^j q_3^k &\rightarrow \varepsilon_{ijk} U^i{}_{i'} U^j{}_{j'} U^k{}_{k'} q_1^{i'} q_2^{j'} q_3^{k'} = \det U \cdot \varepsilon_{i'j'k'} q_1^{i'} q_2^{j'} q_3^{k'} \\ &= \varepsilon_{ijk} q_1^i q_2^j q_3^k \end{aligned}$$

$$\varepsilon_{ijk} \rightarrow \varepsilon_{i'j'k'} U^i{}_{i'} U^j{}_{j'} U^k{}_{k'} = \det U^+ \cdot \varepsilon_{ijk} = \varepsilon_{ijk}$$

Antibaryon ($N_c = 3$)

$$\varepsilon^{ijk} \bar{q}_{1i} \bar{q}_{2j} \bar{q}_{3k} \rightarrow \varepsilon^{ijk} \bar{q}_{1i} \bar{q}_{2j} \bar{q}_{3k}$$

$$\varepsilon^{ijk} \rightarrow \varepsilon^{ijk}$$

Infinitesimal transformations

$$U = 1 + i\alpha^a t^a$$

t^a — generators of the fundamental representation

Infinitesimal transformations

$$U = 1 + i\alpha^a t^a$$

t^a — generators of the fundamental representation

$$U^+ U = 1 + i\alpha^a (t^a - (t^a)^+) = 1 \quad \Rightarrow \quad (t^a)^+ = t^a$$

$$\det U = 1 + i\alpha^a \text{Tr } t^a = 1 \quad \Rightarrow \quad \text{Tr } t^a = 0$$

$$\text{Tr } t^a t^b = T_F \delta^{ab}$$

Infinitesimal transformations

$$U = 1 + i\alpha^a t^a$$

t^a — generators of the fundamental representation

$$U^\dagger U = 1 + i\alpha^a (t^a - (t^a)^\dagger) = 1 \quad \Rightarrow \quad (t^a)^\dagger = t^a$$

$$\det U = 1 + i\alpha^a \text{Tr } t^a = 1 \quad \Rightarrow \quad \text{Tr } t^a = 0$$

$$\text{Tr } t^a t^b = T_F \delta^{ab}$$

Space

Dimension

hermitian matrices

N_c^2

traceless hermitian matrices

$N_c^2 - 1$

$N_c^2 - 1$ generators t^a form a basis

Infinitesimal transformations

$$U = 1 + i\alpha^a t^a$$

t^a — generators of the fundamental representation

$$U^\dagger U = 1 + i\alpha^a (t^a - (t^a)^\dagger) = 1 \quad \Rightarrow \quad (t^a)^\dagger = t^a$$

$$\det U = 1 + i\alpha^a \text{Tr} t^a = 1 \quad \Rightarrow \quad \text{Tr} t^a = 0$$

$$\text{Tr} t^a t^b = T_F \delta^{ab}$$

Space

Dimension

hermitian matrices

N_c^2

traceless hermitian matrices

$N_c^2 - 1$

$N_c^2 - 1$ generators t^a form a basis

$[t^a, t^b]$ is antihermitian and traceless

$$[t^a, t^b] = i f^{abc} t^c \quad i f^{abc} = \frac{1}{T_F} \text{Tr}[t^a, t^b] t^c$$

f^{abc} — structure constants

Adjoint representation

$$A^a = \bar{q} t^a q'$$

$$A^a \rightarrow \bar{q} U^+ t^a U q' = U^{ab} A^b$$

$$U^+ t^a U = U^{ab} t^b$$

$$U^{ab} = \frac{1}{T_F} \text{Tr} U^+ t^a U t^b$$

Adjoint representation

$$A^a = \bar{q} t^a q'$$

$$A^a \rightarrow \bar{q} U^+ t^a U q' = U^{ab} A^b$$

$$U^+ t^a U = U^{ab} t^b$$

$$U^{ab} = \frac{1}{T_F} \text{Tr} U^+ t^a U t^b$$

Invariant tensor $(t^a)^i_j$

$$(t^a)^i_j \rightarrow U^{ab} U^{i'}_i U_j^{j'} (t^b)^{i'}_{j'} = (t^a)^i_j$$

Infinitesimal transformations

$$A^a \rightarrow \bar{q}(1 - i\alpha^c t^c)t^a(1 + i\alpha^c t^c)q' = \bar{q}(t^a + i\alpha^c i f^{acb} t^b)q'$$

$$U^{ab} = \delta^{ab} + i\alpha^c (t^c)^{ab}$$

$$(t^c)^{ab} = i f^{acb}$$

Infinitesimal transformations

$$A^a \rightarrow \bar{q}(1 - i\alpha^c t^c)t^a(1 + i\alpha^c t^c)q' = \bar{q}(t^a + i\alpha^c i f^{acb} t^b)q'$$

$$U^{ab} = \delta^{ab} + i\alpha^c (t^c)^{ab}$$

$$(t^c)^{ab} = i f^{acb}$$

$$(t^a)^{dc}(t^b)^{ce} - (t^b)^{dc}(t^a)^{ce} = i f^{abc}(t^c)^{de}$$

Infinitesimal transformations

$$A^a \rightarrow \bar{q}(1 - i\alpha^c t^c)t^a(1 + i\alpha^c t^c)q' = \bar{q}(t^a + i\alpha^c i f^{acb} t^b)q'$$

$$U^{ab} = \delta^{ab} + i\alpha^c (t^c)^{ab}$$

$$(t^c)^{ab} = i f^{acb}$$

$$(t^a)^{dc}(t^b)^{ce} - (t^b)^{dc}(t^a)^{ce} = i f^{abc}(t^c)^{de}$$

Jacobi identity

$$\begin{aligned} [t^a, [t^b, t^d]] + [t^b, [t^d, t^a]] + [t^d, [t^a, t^b]] &= 0 \\ &= (i f^{bdc} i f^{ace} + i f^{dac} i f^{bce} + i f^{abc} i f^{dce}) t^e \end{aligned}$$

Gluon field

Free quark field

$$L = \bar{q}(i\gamma^\mu \partial_\mu - m)q$$

Invariant with respect to $q(x) \rightarrow Uq(x)$

Gluon field

Free quark field

$$L = \bar{q}(i\gamma^\mu \partial_\mu - m)q$$

Invariant with respect to $q(x) \rightarrow Uq(x)$

How to make it invariant for $q(x) \rightarrow U(x)q(x)$?

$$\partial_\mu q \Rightarrow D_\mu q$$

$$D_\mu q = (\partial_\mu - igA_\mu)q \quad A_\mu = A_\mu^a t^a$$

Gluon field

Free quark field

$$L = \bar{q}(i\gamma^\mu \partial_\mu - m)q$$

Invariant with respect to $q(x) \rightarrow Uq(x)$

How to make it invariant for $q(x) \rightarrow U(x)q(x)$?

$$\partial_\mu q \Rightarrow D_\mu q$$

$$D_\mu q = (\partial_\mu - igA_\mu)q \quad A_\mu = A_\mu^a t^a$$

$$q \rightarrow q' = Uq \quad A_\mu \rightarrow A'_\mu \quad D_\mu q \rightarrow D'_\mu q' = UD_\mu q$$

$$(\partial_\mu - igA'_\mu)Uq = U(\partial_\mu - igA_\mu)q$$

$$\partial_\mu U - igA'_\mu U = -igUA_\mu$$

$$A'_\mu = UA_\mu U^{-1} - \frac{i}{g}(\partial_\mu U)U^{-1}$$

Gluon field strength

Infinitesimal transformation

$$q(x) \rightarrow q'(x) = (1 + i\alpha^a(x)t^a)a(x)$$

$$A_\mu^a(x) \rightarrow A_\mu'^a(x) = A_\mu^a(x) + \frac{1}{g}D_\mu^{ab}\alpha^b(x)$$

$$D_\mu^{ab} = \delta^{ab}\partial_\mu - ig(t^c)^{ab}A_\mu^c$$

Gluon field strength

Infinitesimal transformation

$$q(x) \rightarrow q'(x) = (1 + i\alpha^a(x)t^a)a(x)$$

$$A_\mu^a(x) \rightarrow A_\mu'^a(x) = A_\mu^a(x) + \frac{1}{g}D_\mu^{ab}\alpha^b(x)$$

$$D_\mu^{ab} = \delta^{ab}\partial_\mu - ig(t^c)^{ab}A_\mu^c$$

$$[D_\mu, D_\nu]q$$

$$\begin{aligned} &= \partial_\mu\partial_\nu q - ig(\partial_\mu A_\nu)q - igA_\nu\partial_\mu q - igA_\mu\partial_\nu q - g^2A_\mu A_\nu q \\ &- \partial_\nu\partial_\mu q + ig(\partial_\nu A_\mu)q + igA_\mu\partial_\nu q + igA_\nu\partial_\mu q + g^2A_\nu A_\mu q \\ &= -igG_{\mu\nu}q \end{aligned}$$

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] = G_{\mu\nu}^a t^a$$

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$$

Gluon field strength

Infinitesimal transformation

$$q(x) \rightarrow q'(x) = (1 + i\alpha^a(x)t^a)a(x)$$

$$A_\mu^a(x) \rightarrow A_\mu'^a(x) = A_\mu^a(x) + \frac{1}{g}D_\mu^{ab}\alpha^b(x)$$

$$D_\mu^{ab} = \delta^{ab}\partial_\mu - ig(t^c)^{ab}A_\mu^c$$

$$[D_\mu, D_\nu]q$$

$$\begin{aligned} &= \partial_\mu\partial_\nu q - ig(\partial_\mu A_\nu)q - igA_\nu\partial_\mu q - igA_\mu\partial_\nu q - g^2 A_\mu A_\nu q \\ &- \partial_\nu\partial_\mu q + ig(\partial_\nu A_\mu)q + igA_\mu\partial_\nu q + igA_\nu\partial_\mu q + g^2 A_\nu A_\mu q \\ &= -igG_{\mu\nu}q \end{aligned}$$

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] = G_{\mu\nu}^a t^a$$

$$G_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + gf^{abc}A_\mu^b A_\nu^c$$

$$G_{\mu\nu} \rightarrow UG_{\mu\nu}U^{-1} \quad G_{\mu\nu}^a \rightarrow U^{ab}G_{\mu\nu}^b$$

QCD Lagrangian

$$L = L_q + L_A$$

$$L_q = \sum_f \bar{q}_f (i\gamma^\mu D_\mu + m_f) q_f$$

$$L_A = -\frac{1}{2} \text{Tr} G_{\mu\nu} G^{\mu\nu} = -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu}$$

Flavour symmetries

$U(1): q_f \rightarrow e^{i\alpha} q_f \approx (1 + i\alpha)q_f$
 \Rightarrow baryon number conservation

Flavour symmetries

$U(1)$: $q_f \rightarrow e^{i\alpha} q_f \approx (1 + i\alpha) q_f$
 \Rightarrow baryon number conservation

$$m_f = m$$

$$q_f \rightarrow U_{ff'} q_{f'} \quad U^\dagger U = 1$$

$$U(n_f) = U(1) \times SU(n_f)$$

$$U = e^{i\alpha} U_0 \quad \det U_0 = 1$$

Infinitesimal transformation

$$U = 1 + i\alpha + i\alpha^a \tau^a \quad \text{Tr } \tau^a = 0$$

Chiral symmetries

$$m_f = 0$$

$$q_f = q_{Lf} + q_{Rf} \quad q_{L,R} = \frac{1 \pm \gamma_5}{2} q \quad \gamma_5 q_{L,R} = \pm q_{L,R}$$

$$L = \sum_f \bar{q}_{Lf} i \gamma^\mu D_\mu q_{Lf} + \sum_f \bar{q}_{Rf} i \gamma^\mu D_\mu q_{Rf}$$

Chiral symmetries

$$m_f = 0$$

$$q_f = q_{Lf} + q_{Rf} \quad q_{L,R} = \frac{1 \pm \gamma_5}{2} q \quad \gamma_5 q_{L,R} = \pm q_{L,R}$$

$$L = \sum_f \bar{q}_{Lf} i \gamma^\mu D_\mu q_{Lf} + \sum_f \bar{q}_{Rf} i \gamma^\mu D_\mu q_{Rf}$$

$$U(n_f)_L \times U(n_f)_R$$

$$q_L \rightarrow (1 + i\alpha_L + i\alpha_L^a \tau^a) q_L$$

$$q_R \rightarrow (1 + i\alpha_R + i\alpha_R^a \tau^a) q_L$$

Chiral symmetries

$$m_f = 0$$

$$q_f = q_{Lf} + q_{Rf} \quad q_{L,R} = \frac{1 \pm \gamma_5}{2} q \quad \gamma_5 q_{L,R} = \pm q_{L,R}$$

$$L = \sum_f \bar{q}_{Lf} i \gamma^\mu D_\mu q_{Lf} + \sum_f \bar{q}_{Rf} i \gamma^\mu D_\mu q_{Rf}$$

$$U(n_f)_L \times U(n_f)_R$$

$$q_L \rightarrow (1 + i\alpha_L + i\alpha_L^a \tau^a) q_L$$

$$q_R \rightarrow (1 + i\alpha_R + i\alpha_R^a \tau^a) q_L$$

$$U(n_f)_V \times U(n_f)_A$$

$$q \rightarrow (1 + i\alpha_V + i\alpha_V^a \tau^a + i\alpha_A \gamma_5 + i\alpha_A^a \tau^a \gamma_5) q$$

Conformal symmetry

Scale symmetry

$$x^\mu \rightarrow \lambda x^\mu \quad A^\mu \rightarrow \lambda^{-1} A^\mu \quad q \rightarrow \lambda^{-3/2} q$$

no dimensional parameters in the Lagrangian

Conformal symmetry

Scale symmetry

$$x^\mu \rightarrow \lambda x^\mu \quad A^\mu \rightarrow \lambda^{-1} A^\mu \quad q \rightarrow \lambda^{-3/2} q$$

no dimensional parameters in the Lagrangian

Inversion

$$x^\mu \rightarrow \frac{x^\mu}{x^2}$$

Conformal symmetry

Scale symmetry

$$x^\mu \rightarrow \lambda x^\mu \quad A^\mu \rightarrow \lambda^{-1} A^\mu \quad q \rightarrow \lambda^{-3/2} q$$

no dimensional parameters in the Lagrangian

Inversion

$$x^\mu \rightarrow \frac{x^\mu}{x^2}$$

Special conformal transformation

inversion \Rightarrow translation \Rightarrow inversion

$$x^\mu \rightarrow \frac{x^\mu + a^\mu x^2}{1 + 2a \cdot x + a^2 x^2}$$

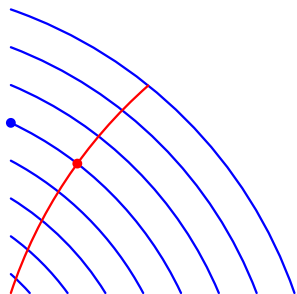
Symmetries

Group	Classical	Quantum
translation Lorentz		
conformal		anomaly
$SU(N_c)$		local
$U(1)$ $SU(n_f)$		
$U(1)_A$ $SU(n_f)_A$		anomaly spontaneously broken
P C T		discrete

Functional integral

$$\begin{aligned}\langle T\{O(x), O(y)\} \rangle &= \frac{\int \prod_{x,a,\mu} dA_{\mu}^a(x) e^{i \int L d^4x} O(x) O(y)}{\int \prod_{x,a,\mu} dA_{\mu}^a(x) e^{i \int L d^4x}} \\ &= \frac{1}{i^2} \frac{\delta^2 Z[j]}{\delta j(x) \delta j(y)} \Big|_{j=0} \\ Z[j] &= \int \prod_{x,a,\mu} dA_{\mu}^a(x) e^{i \int (L+jO) d^4x}\end{aligned}$$

Functional integral for gauge fields



Orbits of the gauge group $A \rightarrow A^U$

Gauge $G^a(A^U(x)) = 0$ — unique solution $U(x)$

for any given $A(x)$

e.g. Lorentz gauge $G^a(A(x)) = \partial^\mu A_\mu^a(x)$

The surface $G = 0$ intersects any orbit at 1 point

Faddeev–Popov determinant

$$\Delta^{-1}[A] = \int \prod_x dU(x) \prod_{x,a} \delta(G^a(A^U(x)))$$

Invariant measure $d(U_0U) = dU$

for infinitesimal transformations $dU = \prod_a d\alpha^a$

Faddeev–Popov determinant

$$\Delta^{-1}[A] = \int \prod_x dU(x) \prod_{x,a} \delta(G^a(A^U(x)))$$

Invariant measure $d(U_0U) = dU$

for infinitesimal transformations $dU = \prod_a d\alpha^a$

Near the surface $G(A^{U_0}) = 0$:

$$\delta G(A(x)) = \hat{M}\alpha(x)$$

$$\Delta^{-1}[A] = \int \prod_x \alpha(x) \delta(\hat{M}\alpha(x)) = 1/\det \hat{M}$$

Faddeev–Popov determinant

$$\Delta^{-1}[A] = \int \prod_x dU(x) \prod_{x,a} \delta(G^a(A^U(x)))$$

Invariant measure $d(U_0U) = dU$

for infinitesimal transformations $dU = \prod_a d\alpha^a$

Near the surface $G(A^{U_0}) = 0$:

$$\delta G(A(x)) = \hat{M}\alpha(x)$$

$$\Delta^{-1}[A] = \int \prod_x \alpha(x) \delta(\hat{M}\alpha(x)) = 1/\det \hat{M}$$

Lorentz gauge $G^a(A(x)) = \partial^\mu A_\mu^a$

$$\delta G^a(x) = \frac{1}{g} \partial^\mu D_\mu^{ab} \alpha^b(x) \Rightarrow \hat{M} = \frac{1}{g} \partial^\mu D_\mu^{ab}$$

Faddeev–Popov determinant is gauge invariant

$$\begin{aligned}\Delta^{-1}[A^{U_0}] &= \int \prod_x dU \prod_{x,a} \delta(G^a(A^{U_0 U}(x))) \\ &= \int \prod_x D(U_0 U) \prod_{x,a} \delta(G^a(A^{U_0 U}(x))) = \Delta^{-1}[A]\end{aligned}$$

Functional integral in a fixed gauge

$$\begin{aligned} Z[J] &= \int \prod_x dA(x) e^{iS[A]} \\ &= \int \prod_x dU(x) \prod_x dA(x) \Delta[A] \prod_x \delta(G(A^U(x))) e^{iS[A]} \\ &= \left(\prod_x \int dU \right) \times \int \prod_x dA(x) \Delta[A] \prod_x \delta(G(A(x))) e^{iS[A]} \end{aligned}$$

Gauss integrals

Bosonic

$$\int dz^* dz e^{-az^*z} \sim \frac{1}{a}$$
$$\int \prod_i dz_i^* dz_i e^{-M_{ij}z_i^*z_j} \sim \frac{1}{\det M}$$

Gauss integrals

Bosonic

$$\int dz^* dz e^{-az^*z} \sim \frac{1}{a}$$
$$\int \prod_i dz_i^* dz_i e^{-M_{ij}z_i^*z_j} \sim \frac{1}{\det M}$$

Fermionic

$$\int dc^* dc e^{-ac^*c} \sim a$$
$$\int \prod_i dc_i^* dc_i e^{-M_{ij}c_i^*c_j} \sim \det M$$

Ghost field

Generalized Lorentz gauge $G^a(A(x)) = \partial^\mu A_\mu^a(x) - \omega^a(x)$

$$\Delta = \det \hat{M} \sim \det \partial^\mu D_\mu^{ab} = \int \prod_{x,a} d\bar{c}^a(x) dc^a(x) e^{i \int L_c d^4x}$$

$$L_c = -\bar{c}^a \partial^\mu D_\mu^{ab} c^b \Rightarrow (\partial^\mu \bar{c}^a) D_\mu^{ab} c^b$$

Ghost field

Generalized Lorentz gauge $G^a(A(x)) = \partial^\mu A_\mu^a(x) - \omega^a(x)$

$$\Delta = \det \hat{M} \sim \det \partial^\mu D_\mu^{ab} = \int \prod_{x,a} d\bar{c}^a(x) dc^a(x) e^{i \int L_c d^4x}$$

$$L_c = -\bar{c}^a \partial^\mu D_\mu^{ab} c^b \Rightarrow (\partial^\mu \bar{c}^a) D_\mu^{ab} c^b$$

$$Z[J] = \int \prod_{x,a} dA^a(x) d\bar{c}^a(x) dc^a(x) \\ \times \prod_{x,a} \delta(\partial^\mu A_\mu^a(x) - \omega^a(x)) e^{i \int (L_A + L_c + J O) d^4x}$$

Ghost field

Generalized Lorentz gauge $G^a(A(x)) = \partial^\mu A_\mu^a(x) - \omega^a(x)$

$$\Delta = \det \hat{M} \sim \det \partial^\mu D_\mu^{ab} = \int \prod_{x,a} d\bar{c}^a(x) dc^a(x) e^{i \int L_c d^4x}$$

$$L_c = -\bar{c}^a \partial^\mu D_\mu^{ab} c^b \Rightarrow (\partial^\mu \bar{c}^a) D_\mu^{ab} c^b$$

$$\begin{aligned} Z[J] &= \int \prod_{x,a} dA^a(x) d\bar{c}^a(x) dc^a(x) \\ &\times \prod_{x,a} \delta(\partial^\mu A_\mu^a(x) - \omega^a(x)) e^{i \int (L_A + L_c + JO) d^4x} \\ &\times \prod_{x,a} d\omega^a(x) e^{-\frac{i}{2a} \int \omega^a(x) \omega^a(x) d^4x} \end{aligned}$$

QCD generating functional

$$Z[J] = \int \prod_{x,a} dA^a(x) d\bar{c}^a(x) dc^a(x) e^{i \int (L+JO) d^4x}$$

$$L = L_A + L_c + L_f$$

$$L_A = -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu}$$

$$L_c = (\partial^\mu \bar{c}^a) D_\mu^{ab} c^b$$

$$L_f = -\frac{1}{2a} (\partial^\mu A_\mu^a)^2$$

QCD generating functional

$$Z[J] = \int \prod_{x,a} dA^a(x) d\bar{c}^a(x) dc^a(x) e^{i \int (L+JO) d^4x}$$

$$L = L_A + L_c + L_f$$

$$L_A = -\frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu}$$

$$L_c = (\partial^\mu \bar{c}^a) D_\mu^{ab} c^b$$

$$L_f = -\frac{1}{2a} (\partial^\mu A_\mu^a)^2$$

QED: ghost does not interact

BRST symmetry

$$\begin{aligned}\delta A_\mu^a &= \lambda^+ D_\mu^{ab} c^b \\ \delta \bar{c}^a &= -\frac{1}{a} \lambda^+ \partial^\mu A_\mu^a \\ \delta c^a &= -\frac{g}{2} f^{abc} \lambda^+ c^b c^c\end{aligned}$$

λ — a constant infinitesimal fermionic parameter

Feynman rules

$$\begin{array}{c} \bullet \longrightarrow \bullet \\ p \end{array} = iS_0(p) \quad S_0(p) = \frac{1}{\not{p}} = \frac{\not{p}}{p^2}$$

$$\begin{array}{c} a \quad b \\ \mu \quad \nu \\ \bullet \text{---} \text{---} \text{---} \text{---} \bullet \\ p \end{array} = -i\delta^{ab}D_{\mu\nu}^0(p)$$

$$\begin{array}{c} a \quad b \\ \bullet \text{---} \text{---} \text{---} \text{---} \bullet \\ p \end{array} = i\delta^{ab}G_0(p) \quad G_0(p) = \frac{1}{p^2}$$

Feynman rules

$$\begin{array}{c} \bullet \longrightarrow \bullet \\ p \end{array} = iS_0(p) \quad S_0(p) = \frac{1}{\not{p}} = \frac{\not{p}}{p^2}$$

$$\begin{array}{c} a \quad b \\ \mu \quad \nu \\ \bullet \text{---} \text{---} \text{---} \bullet \\ p \end{array} = -i\delta^{ab}D_{\mu\nu}^0(p)$$

$$\begin{array}{c} a \quad b \\ \bullet \text{---} \text{---} \bullet \\ p \end{array} = i\delta^{ab}G_0(p) \quad G_0(p) = \frac{1}{p^2}$$

$$\begin{array}{c} \mu \quad a \\ \text{---} \bullet \text{---} \\ | \\ \text{---} \bullet \text{---} \end{array} = t^a \times ig_0\gamma^\mu$$

Feynman rules

$$\begin{array}{c} \bullet \longrightarrow \bullet \\ p \end{array} = iS_0(p) \quad S_0(p) = \frac{1}{\not{p}} = \frac{\not{p}}{p^2}$$

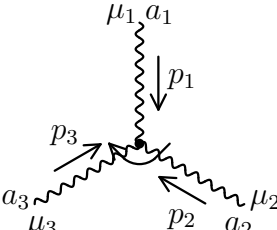
$$\begin{array}{c} a \\ \mu \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \begin{array}{c} b \\ \nu \end{array} = -i\delta^{ab} D_{\mu\nu}^0(p)$$

$$\begin{array}{c} a \\ \bullet \text{---} \bullet \\ p \end{array} = i\delta^{ab} G_0(p) \quad G_0(p) = \frac{1}{p^2}$$

$$\begin{array}{c} \mu \quad a \\ \text{---} \\ \bullet \\ \text{---} \longrightarrow \longrightarrow \end{array} = t^a \times ig_0 \gamma^\mu$$

$$D_{\mu\nu}^0(p) = \frac{1}{p^2} \left[g_{\mu\nu} - (1 - a_0) \frac{p_\mu p_\nu}{p^2} \right]$$

Feynman rules

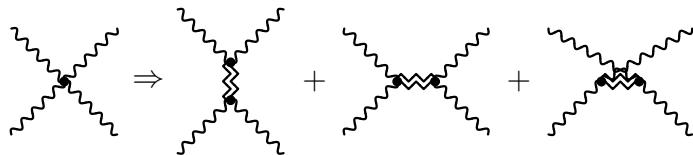


The diagram shows a central vertex where three gluons meet. The top gluon is incoming with momentum p_1 and index μ_1 , and color a_1 . The bottom-left gluon is outgoing with momentum p_3 and index μ_3 , and color a_3 . The bottom-right gluon is outgoing with momentum p_2 and index μ_2 , and color a_2 . The vertex is represented by a small circle with three wavy lines meeting at a point.

$$= i f^{a_1 a_2 a_3} \times i g_0 V^{\mu_1 \mu_2 \mu_3}(p_1, p_2, p_3)$$

$$V^{\mu_1 \mu_2 \mu_3}(p_1, p_2, p_3) =$$
$$(p_3 - p_2)^{\mu_1} g^{\mu_2 \mu_3} + (p_1 - p_3)^{\mu_2} g^{\mu_3 \mu_1} + (p_2 - p_1)^{\mu_3} g^{\mu_1 \mu_2}$$

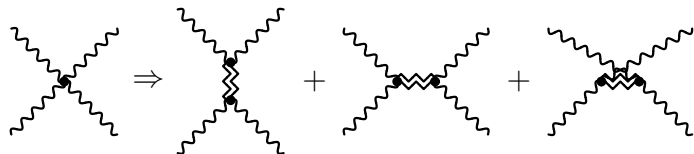
Feynman rules



$$a \begin{array}{c} \nu \\ \bullet \\ \mu \end{array} \begin{array}{c} \beta \\ \bullet \\ \alpha \end{array} b = i\delta^{ab}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})$$

$$b \begin{array}{c} \nu \\ \bullet \\ \mu \end{array} \begin{array}{c} \beta \\ \bullet \\ \alpha \end{array} c = if^{abc} \times g_0 g^{\mu\alpha} g^{\nu\beta}$$

Feynman rules

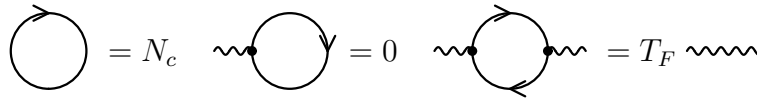


$$a \begin{array}{c} \nu \\ \bullet \\ \mu \end{array} \begin{array}{c} \beta \\ \bullet \\ \alpha \end{array} b = i\delta^{ab}(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})$$

$$b \begin{array}{c} \nu \\ \bullet \\ \mu \end{array} \begin{array}{c} \beta \\ \bullet \\ \alpha \end{array} c = if^{abc} \times g_0 g^{\mu\alpha} g^{\nu\beta}$$

$$c \begin{array}{c} \mu \\ \bullet \\ p \end{array} a = if^{abc} \times ig_0 p^\mu$$

Colour factors

$$\begin{array}{ccc} \text{Tr } 1 = N_c & \text{Tr } t^a = 0 & \text{Tr } t^a t^b = T_F \delta^{ab} \end{array}$$


The image shows three Feynman diagrams representing color factors. The first is a circle with a single arrow pointing clockwise, representing the trace of the identity matrix. The second is a circle with a wavy line entering from the left and an arrow pointing clockwise, representing the trace of a single generator. The third is a circle with two wavy lines entering from the left and two arrows pointing clockwise, representing the trace of the product of two generators.

$$T_F = \frac{1}{2}$$

Cvitanović algorithm

$$(t^a)^i_j (t^a)^k_l = a [\delta_l^i \delta_j^k - b \delta_j^i \delta_l^k]$$

Cvitanović algorithm

$$(t^a)^i_j (t^a)^k_l = a [\delta_l^i \delta_j^k - b \delta_j^i \delta_l^k]$$

Multiply by δ_i^j

$$(t^a)^i_i (t^a)^k_l = 0 = a [\delta_l^k - b N_c \delta_l^k]$$

$$b = \frac{1}{N_c}$$

Cvitanović algorithm

$$(t^a)^i_j (t^a)^k_l = a [\delta_l^i \delta_j^k - b \delta_j^i \delta_l^k]$$

Multiply by δ_i^j

$$(t^a)^i_i (t^a)^k_l = 0 = a [\delta_l^k - b N_c \delta_l^k]$$

$$b = \frac{1}{N_c}$$

Multiply by $(t^b)^j_i$

$$(t^b)^j_i (t^a)^i_j (t^a)^k_l = T_F (t^b)^k_l = a \left[(t^b)^k_l - \frac{1}{N_c} (t^b)^i_i \delta_l^k \right]$$

$$a = T_F$$

Cvitanović algorithm

$$(t^a)^i_j (t^a)^k_l = a [\delta_l^i \delta_j^k - b \delta_j^i \delta_l^k]$$

Multiply by δ_i^j

$$(t^a)^i_i (t^a)^k_l = 0 = a [\delta_l^k - b N_c \delta_l^k]$$

$$b = \frac{1}{N_c}$$

Multiply by $(t^b)^j_i$

$$(t^b)^j_i (t^a)^i_j (t^a)^k_l = T_F (t^b)^k_l = a \left[(t^b)^k_l - \frac{1}{N_c} (t^b)^i_i \delta_l^k \right]$$

$$a = T_F$$

$$(t^a)^i_j (t^a)^k_l = T_F \left[\delta_l^i \delta_j^k - \frac{1}{N_c} \delta_j^i \delta_l^k \right]$$

The image shows a Feynman diagram on the left representing a four-point vertex. Two horizontal lines enter from the left and two exit to the right. A vertical wavy line connects the two upper vertices and the two lower vertices. This diagram is equated to a sum of two diagrams enclosed in large square brackets. The first diagram is multiplied by a coefficient a and the second by $-b$. The first diagram consists of two horizontal lines with a vertical line connecting them, featuring a downward-pointing arrow on the vertical line. The second diagram consists of two horizontal lines with a vertical line connecting them, featuring an upward-pointing arrow on the vertical line.

$$\text{Diagram} = a \left[\text{Diagram 1} \right] - b \left[\text{Diagram 2} \right]$$

The diagram shows the calculation of the ghost-gluon vertex correction. The left side shows the tree-level vertex (a wavy line between two horizontal lines) and its one-loop correction (a ghost loop on the wavy line). The right side shows the corresponding ghost loop diagrams in a bracketed sum, with a minus sign and the parameter b indicating the ghost loop contribution.

$$\begin{aligned}
 & \text{Tree-level vertex} + \text{One-loop correction} \\
 &= a \left[\text{Diagram 1} - b \text{Diagram 2} \right] \\
 &= a \left[\text{Diagram 3} - b \text{Diagram 4} \right] = 0
 \end{aligned}$$

$$b = \frac{1}{N_c}$$

$$\begin{aligned}
 & \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} = a \left[\begin{array}{c} \text{Diagram 3} \quad -b \\ \text{Diagram 4} \end{array} \right] \\
 & \begin{array}{c} \text{Diagram 5} \\ \text{Diagram 6} \end{array} = a \left[\begin{array}{c} \text{Diagram 7} \quad -b \\ \text{Diagram 8} \end{array} \right] = 0
 \end{aligned}$$

$$b = \frac{1}{N_c}$$

$$\begin{array}{c} \text{Diagram 9} \\ \text{Diagram 10} \end{array} = a \left[\begin{array}{c} \text{Diagram 11} \quad -\frac{1}{N_c} \\ \text{Diagram 12} \end{array} \right] = T_F \begin{array}{c} \text{Diagram 13} \\ \text{Diagram 14} \end{array}$$

$$a = T_F$$

Cvitanović algorithm

$$\text{Diagram} = T_F \left[\text{Matrix 1} \text{Matrix 2} - \frac{1}{N_c} \text{Diagram 3} \right]$$

Cvitanović algorithm

$$= T_F \left[\text{square loop with down arrow} + \text{square loop with up arrow} - \frac{1}{N_c} \text{rectangle with arrows} \right]$$

Counting gluons

$$N_g = \text{wavy circle} = \frac{1}{T_F} \text{wavy circle with fermion loop} + \frac{1}{T_F} \text{circle with wavy line} - \frac{1}{N_c} \text{circle} = N_c^2 - 1$$

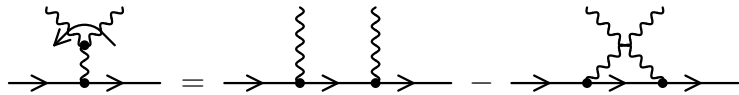
C_F

$$\text{Diagram} = T_F \left[\text{Diagram with loop} - \frac{1}{N_c} \text{Diagram} \right]$$

$$= T_F \left(N_c - \frac{1}{N_c} \right) \longrightarrow$$

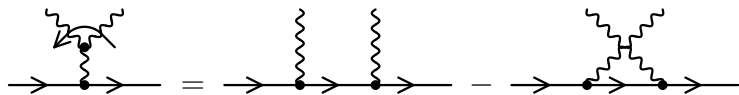
$$t^a t^a = C_F \quad C_F = T_F \left(N_c - \frac{1}{N_c} \right)$$

3-gluon vertex

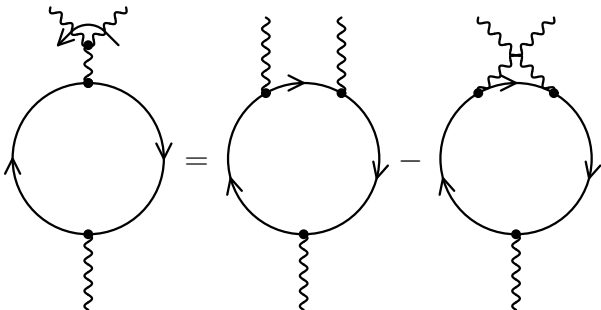


$$[t^a, t^b] = i f^{abc} t^c$$

3-gluon vertex



$$[t^a, t^b] = i f^{abc} t^c$$



3-gluon vertex



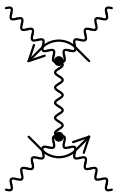
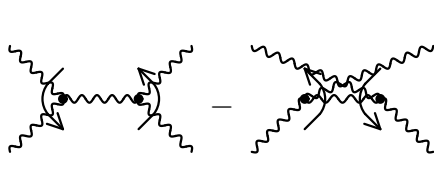
The diagram on the left shows a 3-gluon vertex. It consists of a central vertex with three wavy lines (gluons) attached. Two wavy lines enter from the top, and one wavy line exits from the bottom. The vertex is represented by a small semi-circular loop with a red dot at its center.

$$= \frac{1}{T_F} \left[\text{Diagram 1} - \text{Diagram 2} \right]$$

The diagram on the right shows the decomposition of the 3-gluon vertex into two loop diagrams. The first diagram is a circular loop with three wavy lines attached at the top, left, and right positions. The second diagram is a circular loop with three wavy lines attached at the top, left, and bottom positions. The two diagrams are subtracted from each other.

3-gluon vertex


$$= \frac{1}{T_F} \left[\text{Diagram 1} - \text{Diagram 2} \right]$$



$$= \text{Diagram 3} - \text{Diagram 4}$$


C_A

The diagram shows a ghost loop with a wavy line representing a ghost and a solid line representing a fermion. The ghost loop is a circle with a wavy line on the left and a wavy line on the right. The fermion loop is a circle with a solid line on the top and a solid line on the bottom. The two loops are connected by a wavy line in the middle.

$$\begin{aligned} &= \frac{2}{T_F^2} \left[\text{Diagram 1} - \text{Diagram 2} \right] \\ &= \frac{2}{T_F} \left[\text{Diagram 3} - \frac{1}{N_c} \text{Diagram 4} \right] \\ &\quad - \left[\text{Diagram 5} + \frac{1}{N_c} \text{Diagram 6} \right] \end{aligned}$$

The diagrams are as follows:

- Diagram 1: Two fermion loops connected by a wavy line. The top fermion line has arrows pointing right, and the bottom fermion line has arrows pointing left.
- Diagram 2: Two fermion loops connected by a wavy line. The top fermion line has arrows pointing left, and the bottom fermion line has arrows pointing right.
- Diagram 3: A single fermion loop with a wavy line attached to the bottom. The fermion line has an arrow pointing right.
- Diagram 4: Two fermion loops connected by a wavy line. The top fermion line has an arrow pointing right, and the bottom fermion line has an arrow pointing left.
- Diagram 5: A single fermion loop with a wavy line attached to the bottom. The fermion line has an arrow pointing left.
- Diagram 6: Two fermion loops connected by a wavy line. The top fermion line has an arrow pointing left, and the bottom fermion line has an arrow pointing right.

C_A

$$\begin{aligned} &= \frac{2}{T_F} \left[\text{Diagram 1} - \text{Diagram 2} \right] \\ &= 2 \left[\text{Diagram 3} - \frac{1}{N_c} \text{Diagram 4} \right. \\ &\quad \left. - \text{Diagram 5} + \frac{1}{N_c} \text{Diagram 6} \right] \\ &= 2T_F N_c \text{Diagram 7} \end{aligned}$$

The diagrams are as follows:

- Diagram 1: A circle with a wavy line (gluon) in the middle, connected to the circle at two points. The circle has arrows indicating a clockwise flow.
- Diagram 2: A circle with a wavy line (gluon) in the middle, connected to the circle at two points. The circle has arrows indicating a clockwise flow.
- Diagram 3: A circle with a wavy line (gluon) on the left and right sides, connected to the circle at two points. The circle has arrows indicating a clockwise flow.
- Diagram 4: A circle with a wavy line (gluon) on the left and right sides, connected to the circle at two points. The circle has arrows indicating a clockwise flow.
- Diagram 5: A circle with a wavy line (gluon) on the left and right sides, connected to the circle at two points. The circle has arrows indicating a clockwise flow.
- Diagram 6: A circle with a wavy line (gluon) on the left and right sides, connected to the circle at two points. The circle has arrows indicating a clockwise flow.
- Diagram 7: A wavy line (gluon).

C_A

$$i f^{acd} i f^{bdc} = C_A \delta^{ab} \quad C_A = 2T_F N_c$$

Example 1

The diagram shows a horizontal line with three vertices marked by black dots. A wavy gluon line forms a loop between the first and second vertices. This is equal to T_F times the difference between a ghost loop diagram (a loop with two vertices and a wavy gluon line) and a tadpole diagram (a single vertex with a wavy gluon line).

$$\text{Diagram} = T_F \left[\text{Diagram 1} - \frac{1}{N_c} \text{Diagram 2} \right]$$

The diagram shows a horizontal line with one vertex marked by a black dot. A wavy gluon line is attached to this vertex. The coefficient is $-\frac{T_F}{N_c}$.

$$= -\frac{T_F}{N_c} \text{Diagram}$$

$$t^a t^b t^a = -\frac{T_F}{N_c} t^b \quad -\frac{T_F}{N_c} = C_F - \frac{C_A}{2}$$

Example 2

$$\begin{aligned}
 & \text{Diagram 1} = \frac{1}{T_F} \left[\text{Diagram 2} - \text{Diagram 3} \right] \\
 & = \text{Diagram 4} - \frac{1}{N_c} \text{Diagram 5} - \text{Diagram 6} + \frac{1}{N_c} \text{Diagram 7} \\
 & = \text{Diagram 8} - \text{Diagram 9}
 \end{aligned}$$

The diagram shows the decomposition of a triangle diagram into a difference of two circle diagrams, and further into four diagrams with N_c factors, and finally into two diagrams with a wavy line.

Example 2

$$\begin{aligned}
 &= T_F \left[\begin{array}{c} \text{Diagram 1} - \frac{1}{N_c} \text{Diagram 2} \\ - \text{Diagram 3} + \frac{1}{N_c} \text{Diagram 4} \end{array} \right] \\
 &= T_F N_c \text{Diagram 5} \\
 i f^{abc} t^b t^a &= \frac{C_A}{2} t^c
 \end{aligned}$$

The diagrams are:

 Diagram 1: A horizontal line with an arrow pointing right. A wavy line (gluon) is attached to the line at a black dot. The wavy line forms a loop that connects back to the line at another black dot.

 Diagram 2: A horizontal line with an arrow pointing right. A wavy line is attached to the line at a black dot and extends to the right.

 Diagram 3: A horizontal line with an arrow pointing right. A wavy line is attached to the line at a black dot and forms a loop that connects back to the line at another black dot.

 Diagram 4: A horizontal line with an arrow pointing right. A wavy line is attached to the line at a black dot and extends to the right.

 Diagram 5: A horizontal line with an arrow pointing right. A wavy line is attached to the line at a black dot and extends to the right.

Shorter solution

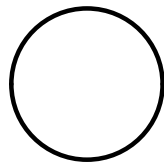
$$\begin{aligned} & \text{Diagram 1} = \frac{1}{2} \left[\text{Diagram 2} - \text{Diagram 3} \right] \\ & = \frac{1}{2} \text{Diagram 4} = \frac{C_A}{2} \text{Diagram 5} \end{aligned}$$

The diagrams are as follows:

- Diagram 1:** A fermion line with two vertices. The upper vertex is a triangle loop with a fermion line and a wavy line. A wavy line is attached to the top vertex.
- Diagram 2:** Similar to Diagram 1, but the wavy line is attached to the top vertex with an arrow pointing away from the loop.
- Diagram 3:** Similar to Diagram 1, but the wavy line is attached to the top vertex with an arrow pointing towards the loop.
- Diagram 4:** A fermion line with two vertices. The upper vertex is a box loop with a fermion line and a wavy line. A wavy line is attached to the top vertex.
- Diagram 5:** A fermion line with two vertices. A wavy line is attached to the top vertex.

1-loop massive vacuum diagram

We are going to live in $d = 4 - 2\varepsilon$ dimensional space-time

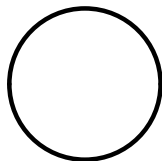


$$\int \frac{d^d k}{D^n} = i\pi^{d/2} m^{d-2n} V(n)$$

$$D = m^2 - k^2 - i0$$

1-loop massive vacuum diagram

We are going to live in $d = 4 - 2\varepsilon$ dimensional space-time



$$\int \frac{d^d k}{D^n} = i\pi^{d/2} m^{d-2n} V(n)$$

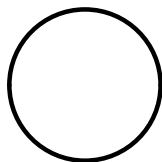
$$D = m^2 - k^2 - i0$$

Wick rotation $k_0 = ik_0$, $k^2 = -k^2$

$$\int \frac{d^d k}{(k^2 + 1)^n} = \pi^{d/2} V(n)$$

1-loop massive vacuum diagram

We are going to live in $d = 4 - 2\varepsilon$ dimensional space-time



$$\int \frac{d^d k}{D^n} = i\pi^{d/2} m^{d-2n} V(n)$$

$$D = m^2 - k^2 - i0$$

Wick rotation $k_0 = ik_0$, $k^2 = -k^2$

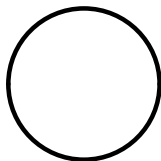
$$\int \frac{d^d k}{(k^2 + 1)^n} = \pi^{d/2} V(n)$$

α parametrization

$$\frac{1}{a^n} = \frac{1}{\Gamma(n)} \int_0^\infty e^{-a\alpha} \alpha^{n-1} d\alpha$$

1-loop massive vacuum diagram

We are going to live in $d = 4 - 2\varepsilon$ dimensional space-time



$$\int \frac{d^d k}{D^n} = i\pi^{d/2} m^{d-2n} V(n)$$

$$D = m^2 - k^2 - i0$$

Wick rotation $k_0 = ik_0$, $k^2 = -k^2$

$$\int \frac{d^d k}{(k^2 + 1)^n} = \pi^{d/2} V(n)$$

α parametrization

$$\frac{1}{a^n} = \frac{1}{\Gamma(n)} \int_0^\infty e^{-a\alpha} \alpha^{n-1} d\alpha$$

$$V(n) = \frac{\pi^{-d/2}}{\Gamma(n)} \int e^{-\alpha(k^2+1)} \alpha^{n-1} d\alpha d^d k$$

1-loop massive vacuum diagram

$$\int e^{-\alpha k^2} d^d \mathbf{k} = \left[\int_{-\infty}^{+\infty} e^{-\alpha k_x^2} dk_x \right]^d = \left(\frac{\pi}{\alpha} \right)^{d/2}$$

1-loop massive vacuum diagram

$$\int e^{-\alpha k^2} d^d \mathbf{k} = \left[\int_{-\infty}^{+\infty} e^{-\alpha k_x^2} dk_x \right]^d = \left(\frac{\pi}{\alpha} \right)^{d/2}$$

$$V(n) = \frac{1}{\Gamma(n)} \int_0^\infty e^{-\alpha} \alpha^{n-d/2-1} d\alpha$$

1-loop massive vacuum diagram

$$\int e^{-\alpha k^2} d^d \mathbf{k} = \left[\int_{-\infty}^{+\infty} e^{-\alpha k_x^2} dk_x \right]^d = \left(\frac{\pi}{\alpha} \right)^{d/2}$$

$$V(n) = \frac{1}{\Gamma(n)} \int_0^\infty e^{-\alpha} \alpha^{n-d/2-1} d\alpha$$

$$V(n) = \frac{\Gamma(-d/2 + n)}{\Gamma(n)}$$

1-loop massive vacuum diagram

$$\int e^{-\alpha k^2} d^d k = \left[\int_{-\infty}^{+\infty} e^{-\alpha k_x^2} dk_x \right]^d = \left(\frac{\pi}{\alpha} \right)^{d/2}$$

$$V(n) = \frac{1}{\Gamma(n)} \int_0^\infty e^{-\alpha} \alpha^{n-d/2-1} d\alpha$$

$$V(n) = \frac{\Gamma(-d/2 + n)}{\Gamma(n)}$$

For integer n , proportional to

$$V_1 = \frac{4}{(d-2)(d-4)} \Gamma(1 + \varepsilon)$$

d -dimensional solid angle

$$\int e^{-k^2} d^d \mathbf{k} = \left[\int_{-\infty}^{+\infty} e^{-k_x^2} dk_x \right]^d = \pi^{d/2}$$

d -dimensional solid angle

$$\begin{aligned}\int e^{-k^2} d^d \mathbf{k} &= \left[\int_{-\infty}^{+\infty} e^{-k_x^2} dk_x \right]^d = \pi^{d/2} \\ &= \Omega_d \int_0^\infty e^{-k^2} k^{d-1} dk = \frac{\Omega_d}{2} \int_0^\infty e^{-k^2} (k^2)^{d/2-1} dk^2 = \frac{\Omega_d \Gamma(d/2)}{2}\end{aligned}$$

d -dimensional solid angle

$$\begin{aligned}\int e^{-k^2} d^d \mathbf{k} &= \left[\int_{-\infty}^{+\infty} e^{-k_x^2} dk_x \right]^d = \pi^{d/2} \\ &= \Omega_d \int_0^\infty e^{-k^2} k^{d-1} dk = \frac{\Omega_d}{2} \int_0^\infty e^{-k^2} (k^2)^{d/2-1} dk^2 = \frac{\Omega_d \Gamma(d/2)}{2}\end{aligned}$$

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

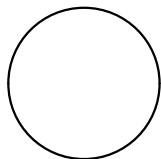
d -dimensional solid angle

$$\begin{aligned}\int e^{-k^2} d^d \mathbf{k} &= \left[\int_{-\infty}^{+\infty} e^{-k_x^2} dk_x \right]^d = \pi^{d/2} \\ &= \Omega_d \int_0^\infty e^{-k^2} k^{d-1} dk = \frac{\Omega_d}{2} \int_0^\infty e^{-k^2} (k^2)^{d/2-1} dk^2 = \frac{\Omega_d \Gamma(d/2)}{2}\end{aligned}$$

$$\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

$$\Omega_1 = 2 \quad \Omega_2 = 2\pi \quad \Omega_3 = 4\pi \quad \Omega_4 = 2\pi^2 \quad \dots$$

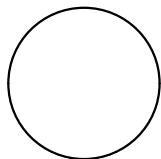
Massless vacuum diagrams



$$\int \frac{d^d k}{(-k^2 - i0)^n} = 0$$

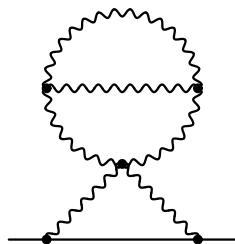
by dimensionality (argument fails at $n = d/2$)

Massless vacuum diagrams



$$\int \frac{d^d k}{(-k^2 - i0)^n} = 0$$

by dimensionality (argument fails at $n = d/2$)



$$= 0$$

Feynman parametrization

α parametrization

$$\frac{1}{a_1^{n_1} a_2^{n_2}} = \frac{1}{\Gamma(n_1)\Gamma(n_2)} \int e^{-a_1\alpha_1 - a_2\alpha_2} \alpha_1^{n_1-1} \alpha_2^{n_2-1} d\alpha_1 d\alpha_2$$

Feynman parametrization

α parametrization

$$\frac{1}{a_1^{n_1} a_2^{n_2}} = \frac{1}{\Gamma(n_1)\Gamma(n_2)} \int e^{-a_1\alpha_1 - a_2\alpha_2} \alpha_1^{n_1-1} \alpha_2^{n_2-1} d\alpha_1 d\alpha_2$$

Substitution $\alpha_1 = \eta x$, $\alpha_2 = \eta(1 - x)$

$$\frac{1}{a_1^{n_1} a_2^{n_2}} = \frac{\Gamma(n_1 + n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 \frac{x^{n_1-1} (1-x)^{n_2-1} dx}{[a_1 x + a_2 (1-x)]^{n_1+n_2}}$$

Feynman parametrization

α parametrization

$$\frac{1}{a_1^{n_1} a_2^{n_2}} = \frac{1}{\Gamma(n_1)\Gamma(n_2)} \int e^{-a_1\alpha_1 - a_2\alpha_2} \alpha_1^{n_1-1} \alpha_2^{n_2-1} d\alpha_1 d\alpha_2$$

Substitution $\alpha_1 = \eta x$, $\alpha_2 = \eta(1 - x)$

$$\frac{1}{a_1^{n_1} a_2^{n_2}} = \frac{\Gamma(n_1 + n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 \frac{x^{n_1-1} (1-x)^{n_2-1} dx}{[a_1 x + a_2 (1-x)]^{n_1+n_2}}$$

Substitution $\alpha_1 = \eta x$, $\alpha_2 = \eta$

$$\frac{1}{a_1^{n_1} a_2^{n_2}} = \frac{\Gamma(n_1 + n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^\infty \frac{x^{n_1-1} dx}{[a_1 x + a_2]^{n_1+n_2}}$$

Feynman parametrization

$$\frac{1}{a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}} = \frac{1}{\Gamma(n_1) \Gamma(n_2) \cdots \Gamma(n_k)} \int e^{-a_1 \alpha_1 - a_2 \alpha_2 \cdots - a_k \alpha_k} \alpha_1^{n_1-1} \alpha_2^{n_2-1} \cdots \alpha_k^{n_k-1} d\alpha_1 d\alpha_2 \cdots d\alpha_k$$

Feynman parametrization

$$\frac{1}{a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}} = \frac{1}{\Gamma(n_1) \Gamma(n_2) \cdots \Gamma(n_k)} \int e^{-a_1 \alpha_1 - a_2 \alpha_2 \cdots - a_k \alpha_k} \alpha_1^{n_1-1} \alpha_2^{n_2-1} \cdots \alpha_k^{n_k-1} d\alpha_1 d\alpha_2 \cdots d\alpha_k \times \delta(\alpha_1 + \alpha_2 \cdots + \alpha_l - \eta) d\eta \quad (1 \leq l \leq k)$$

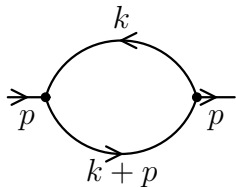
Feynman parametrization

$$\frac{1}{a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}} = \frac{1}{\Gamma(n_1) \Gamma(n_2) \cdots \Gamma(n_k)} \int e^{-a_1 \alpha_1 - a_2 \alpha_2 \cdots - a_k \alpha_k} \alpha_1^{n_1-1} \alpha_2^{n_2-1} \cdots \alpha_k^{n_k-1} d\alpha_1 d\alpha_2 \cdots d\alpha_k \times \delta(\alpha_1 + \alpha_2 \cdots + \alpha_l - \eta) d\eta \quad (1 \leq l \leq k)$$

$$\alpha_i = \eta x_i$$

$$\frac{1}{a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k}} = \frac{\Gamma(n_1 + n_2 \cdots + n_k)}{\Gamma(n_1) \Gamma(n_2) \cdots \Gamma(n_k)} \int \frac{\delta(x_1 + x_2 \cdots + x_l - 1) x_1^{n_1-1} x_2^{n_2-1} \cdots x_k^{n_k-1} dx_1 dx_2 \cdots dx_k}{[a_1 x_1 + a_2 x_2 \cdots + a_k x_k]^{n_1+n_2+\cdots+n_k}}$$

1-loop massless propagator diagram

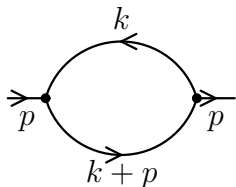


$$\int \frac{d^d k}{D_1^{n_1} D_2^{n_2}} = i\pi^{d/2} (-p^2)^{d/2-n_1-n_2} G(n_1, n_2)$$

$$D_1 = -(k+p)^2 \quad D_2 = -k^2$$

Vanishes for integer $n_1 \leq 0$ or $n_2 \leq 0$

1-loop massless propagator diagram



$$\int \frac{d^d k}{D_1^{n_1} D_2^{n_2}} = i\pi^{d/2} (-p^2)^{d/2-n_1-n_2} G(n_1, n_2)$$

$$D_1 = -(k+p)^2 \quad D_2 = -k^2$$

Vanishes for integer $n_1 \leq 0$ or $n_2 \leq 0$

Wick rotation, α parametrization

$$G(n_1, n_2) = \frac{\pi^{-d/2}}{\Gamma(n_1)\Gamma(n_2)} \times \int e^{-\alpha_1(k+p)^2 - \alpha_2 k^2} \alpha_1^{n_1-1} \alpha_2^{n_2-1} d\alpha_1 d\alpha_2 d^d k$$

1-loop massless propagator diagram

$$\text{Shift } \mathbf{k}' = \mathbf{k} + \frac{\alpha_1}{\alpha_1 + \alpha_2} \mathbf{p}$$

$$\begin{aligned} G(n_1, n_2) &= \frac{\pi^{-d/2}}{\Gamma(n_1)\Gamma(n_2)} \int \exp\left[-\frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2}\right] \alpha_1^{n_1-1} \alpha_2^{n_2-1} d\alpha_1 d\alpha_2 \\ &\quad \times \int e^{-(\alpha_1+\alpha_2)\mathbf{k}^2} d^d \mathbf{k} \\ &= \frac{1}{\Gamma(n_1)\Gamma(n_2)} \\ &\quad \times \int \exp\left[-\frac{\alpha_1\alpha_2}{\alpha_1 + \alpha_2}\right] (\alpha_1 + \alpha_2)^{-d/2} \alpha_1^{n_1-1} \alpha_2^{n_2-1} d\alpha_1 d\alpha_2 \end{aligned}$$

1-loop massless propagator diagram

Substitution $\alpha_1 = \eta x$, $\alpha_2 = \eta(1 - x)$

$$\begin{aligned} G(n_1, n_2) &= \frac{1}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 x^{n_1-1} (1-x)^{n_2-1} dx \\ &\quad \times \int_0^\infty e^{-\eta x(1-x)} \eta^{-d/2+n_1+n_2-1} d\eta \\ &= \frac{\Gamma(-d/2 + n_1 + n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 x^{d/2-n_2-1} (1-x)^{d/2-n_1-1} dx \end{aligned}$$

1-loop massless propagator diagram

Substitution $\alpha_1 = \eta x$, $\alpha_2 = \eta(1 - x)$

$$\begin{aligned} G(n_1, n_2) &= \frac{1}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 x^{n_1-1} (1-x)^{n_2-1} dx \\ &\quad \times \int_0^\infty e^{-\eta x(1-x)} \eta^{-d/2+n_1+n_2-1} d\eta \\ &= \frac{\Gamma(-d/2 + n_1 + n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 x^{d/2-n_2-1} (1-x)^{d/2-n_1-1} dx \end{aligned}$$

$$G(n_1, n_2) = \frac{\Gamma(-d/2 + n_1 + n_2)\Gamma(d/2 - n_1)\Gamma(d/2 - n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(d - n_1 - n_2)}$$

1-loop massless propagator diagram

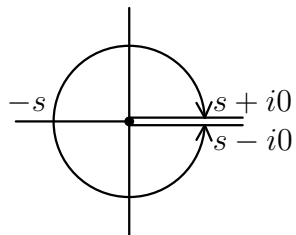
For integer $n_{1,2}$, proportional to

$$G_1 = -\frac{2g_1}{(d-3)(d-4)}$$
$$g_1 = \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

UV divergence if $n_1 + n_2 \leq 2$

IR divergence if $n_1 \geq 2$ or $n_2 \geq 2$

Analytical properties



$$I(s \pm i0) = G_1 s^{-\varepsilon} e^{\pm i\pi\varepsilon}$$

$$I(s + i0) - I(s - i0)$$

$$= G_1 s^{-\varepsilon} 2\varepsilon \sin(\pi\varepsilon) \rightarrow 2\pi i$$

$$I(p^2) = -\frac{i}{\pi^{d/2}} \int \frac{d^d k}{(-k^2 - i0)(-(k+p)^2 - i0)} = G_1 (-p^2)^{-\varepsilon}$$

Massless QCD

$$L = \sum_i \bar{q}_{0i} i \not{D} q_{0i} - \frac{1}{4} G_{0\mu\nu}^a G_0^{a\mu\nu}$$

$$D_\mu q_0 = (\partial_\mu - i g_0 A_{0\mu}) q_0 \quad A_{0\mu} = A_{0\mu}^a t^a$$

$$[D_\mu, D_\nu] q_0 = -i g_0 G_{0\mu\nu} q_0 \quad G_{0\mu\nu} = G_{0\mu\nu}^a t^a$$

$$G_{0\mu\nu}^a = \partial_\mu A_{0\nu}^a - \partial_\nu A_{0\mu}^a + g_0 f^{abc} A_{0\mu}^b A_{0\nu}^c$$

Massless QCD

$$L = \sum_i \bar{q}_{0i} i \not{D} q_{0i} - \frac{1}{4} G_{0\mu\nu}^a G_0^{a\mu\nu} - \frac{1}{2a_0} (\partial_\mu A_0^{a\mu})^2 + (\partial^\mu \bar{c}_0^a) (D_\mu c_0^a)$$

$$D_\mu q_0 = (\partial_\mu - ig_0 A_{0\mu}) q_0 \quad A_{0\mu} = A_{0\mu}^a t^a$$

$$[D_\mu, D_\nu] q_0 = -ig_0 G_{0\mu\nu} q_0 \quad G_{0\mu\nu} = G_{0\mu\nu}^a t^a$$

$$G_{0\mu\nu}^a = \partial_\mu A_{0\nu}^a - \partial_\nu A_{0\mu}^a + g_0 f^{abc} A_{0\mu}^b A_{0\nu}^c$$

$$D_\mu c_0^a = (\partial_\mu \delta^{ab} - ig_0 A_{0\mu}^{ab}) c_0^b \quad A_{0\mu}^{ab} = A_{0\mu}^c (t^c)^{ab}$$

$$[t^a, t^b] = i f^{abc} t^c \quad (t^c)^{ab} = i f^{acb}$$

Renormalization

Renormalized quantities

$$q_0 = Z_q^{1/2} q \quad A_0 = Z_A^{1/2} A \quad a_0 = Z_A a \quad g_0 = Z_\alpha^{1/2} g$$

Renormalization

Renormalized quantities

$$q_0 = Z_q^{1/2} q \quad A_0 = Z_A^{1/2} A \quad a_0 = Z_A a \quad g_0 = Z_\alpha^{1/2} g$$

Minimal renormalization constants

$$Z_i(\alpha_s) = 1 + \frac{z_1}{\varepsilon} \frac{\alpha_s}{4\pi} + \left(\frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon} \right) \left(\frac{\alpha_s}{4\pi} \right)^2 + \dots$$

Renormalization

Renormalized quantities

$$q_0 = Z_q^{1/2} q \quad A_0 = Z_A^{1/2} A \quad a_0 = Z_A a \quad g_0 = Z_\alpha^{1/2} g$$

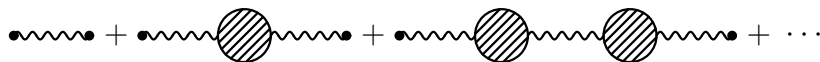
Minimal renormalization constants

$$Z_i(\alpha_s) = 1 + \frac{z_1}{\varepsilon} \frac{\alpha_s}{4\pi} + \left(\frac{z_{22}}{\varepsilon^2} + \frac{z_{21}}{\varepsilon} \right) \left(\frac{\alpha_s}{4\pi} \right)^2 + \dots$$

Dimensionalities: $[L] = d$, $[A_0] = 1 - \varepsilon$, $[q_0] = 3/2 - \varepsilon$,
 $[g_0] = \varepsilon$. Exactly dimensionless

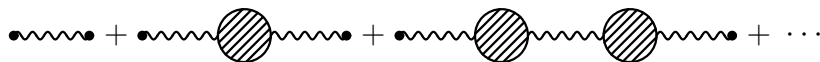
$$\frac{\alpha_s(\mu)}{4\pi} = \mu^{-2\varepsilon} \frac{g^2}{(4\pi)^{d/2}} e^{-\gamma\varepsilon} \quad \frac{g_0^2}{(4\pi)^{d/2}} = \mu^{2\varepsilon} \frac{\alpha_s(\mu)}{4\pi} Z_\alpha(\alpha(\mu)) e^{\gamma\varepsilon}$$

Gluon propagator



$$\begin{aligned} -iD_{\mu\nu}(p) &= -iD_{\mu\nu}^0(p) + (-i)D_{\mu\alpha}^0(p)i\Pi^{\alpha\beta}(p)(-i)D_{\beta\nu}^0(p) \\ &+ (-i)D_{\mu\alpha}^0(p)i\Pi^{\alpha\beta}(p)(-i)D_{\beta\gamma}^0(p)i\Pi^{\gamma\delta}(p)(-i)D_{\gamma\nu}^0(p) + \dots \end{aligned}$$

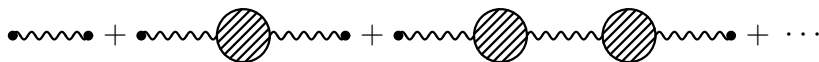
Gluon propagator



$$\begin{aligned} -iD_{\mu\nu}(p) &= -iD_{\mu\nu}^0(p) + (-i)D_{\mu\alpha}^0(p)i\Pi^{\alpha\beta}(p)(-i)D_{\beta\nu}^0(p) \\ &+ (-i)D_{\mu\alpha}^0(p)i\Pi^{\alpha\beta}(p)(-i)D_{\beta\gamma}^0(p)i\Pi^{\gamma\delta}(p)(-i)D_{\gamma\nu}^0(p) + \dots \end{aligned}$$

$$D_{\mu\nu}(p) = D_{\mu\nu}^0(p) + D_{\mu\alpha}^0(p)\Pi^{\alpha\beta}(p)D_{\beta\nu}(p)$$

Gluon propagator

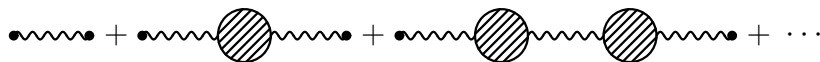


$$\begin{aligned} -iD_{\mu\nu}(p) &= -iD_{\mu\nu}^0(p) + (-i)D_{\mu\alpha}^0(p)i\Pi^{\alpha\beta}(p)(-i)D_{\beta\nu}^0(p) \\ &+ (-i)D_{\mu\alpha}^0(p)i\Pi^{\alpha\beta}(p)(-i)D_{\beta\gamma}^0(p)i\Pi^{\gamma\delta}(p)(-i)D_{\gamma\nu}^0(p) + \dots \end{aligned}$$

$$D_{\mu\nu}(p) = D_{\mu\nu}^0(p) + D_{\mu\alpha}^0(p)\Pi^{\alpha\beta}(p)D_{\beta\nu}(p)$$

$$\left. \begin{aligned} A_{\mu\nu} &= A_{\perp} \left[g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} \right] + A_{\parallel} \frac{p_{\mu}p_{\nu}}{p^2} \\ A_{\mu\nu}^{-1} &= A_{\perp}^{-1} \left[g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} \right] + A_{\parallel}^{-1} \frac{p_{\mu}p_{\nu}}{p^2} \end{aligned} \right\} A_{\mu\lambda}^{-1}A^{\lambda\nu} = \delta_{\mu}^{\nu}$$

Gluon propagator



$$\begin{aligned}
 -iD_{\mu\nu}(p) &= -iD_{\mu\nu}^0(p) + (-i)D_{\mu\alpha}^0(p)i\Pi^{\alpha\beta}(p)(-i)D_{\beta\nu}^0(p) \\
 &+ (-i)D_{\mu\alpha}^0(p)i\Pi^{\alpha\beta}(p)(-i)D_{\beta\gamma}^0(p)i\Pi^{\gamma\delta}(p)(-i)D_{\gamma\nu}^0(p) + \dots
 \end{aligned}$$

$$D_{\mu\nu}(p) = D_{\mu\nu}^0(p) + D_{\mu\alpha}^0(p)\Pi^{\alpha\beta}(p)D_{\beta\nu}(p)$$

$$\left. \begin{aligned}
 A_{\mu\nu} &= A_{\perp} \left[g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} \right] + A_{\parallel} \frac{p_{\mu}p_{\nu}}{p^2} \\
 A_{\mu\nu}^{-1} &= A_{\perp}^{-1} \left[g_{\mu\nu} - \frac{p_{\mu}p_{\nu}}{p^2} \right] + A_{\parallel}^{-1} \frac{p_{\mu}p_{\nu}}{p^2}
 \end{aligned} \right\} A_{\mu\lambda}^{-1}A^{\lambda\nu} = \delta_{\mu}^{\nu}$$

$$D_{\mu\nu}^{-1}(p) = (D^0)_{\mu\nu}^{-1}(p) - \Pi_{\mu\nu}(p)$$

Gluon propagator

Slavnov–Taylor identity $\Pi_{\mu\nu}(p)p^\nu = 0$

$$\Pi_{\mu\nu}(p) = (p^2 g_{\mu\nu} - p_\mu p_\nu)\Pi(p^2)$$

Gluon propagator

Slavnov–Taylor identity $\Pi_{\mu\nu}(p)p^\nu = 0$

$$\Pi_{\mu\nu}(p) = (p^2 g_{\mu\nu} - p_\mu p_\nu)\Pi(p^2)$$

$$D_{\mu\nu}(p) = \frac{1}{p^2(1 - \Pi(p^2))} \left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] + a_0 \frac{p_\mu p_\nu}{(p^2)^2}$$

Gluon propagator

Slavnov–Taylor identity $\Pi_{\mu\nu}(p)p^\nu = 0$

$$\Pi_{\mu\nu}(p) = (p^2 g_{\mu\nu} - p_\mu p_\nu) \Pi(p^2)$$

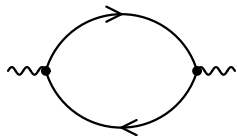
$$D_{\mu\nu}(p) = \frac{1}{p^2(1 - \Pi(p^2))} \left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] + a_0 \frac{p_\mu p_\nu}{(p^2)^2}$$

Renormalized propagator $D_{\mu\nu}(p) = Z_A(\alpha(\mu)) D_{\mu\nu}^r(p; \mu)$

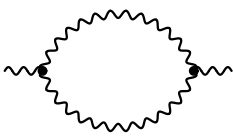
$$D_{\mu\nu}^r(p; \mu) = D_{\perp}^r(p^2; \mu) \left[g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] + a(\mu) \frac{p_\mu p_\nu}{(p^2)^2}$$

Slavnov–Taylor identities

Gluon self-energy

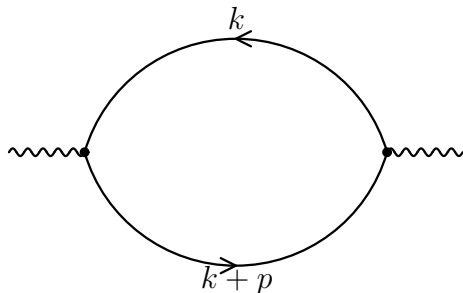


$$i\delta^{ab}\Pi_{\mu\nu}(p)$$



$$\Pi_{\mu\nu}(p) = (p^2 g_{\mu\nu} - p_\mu p_\nu)\Pi(p^2)$$

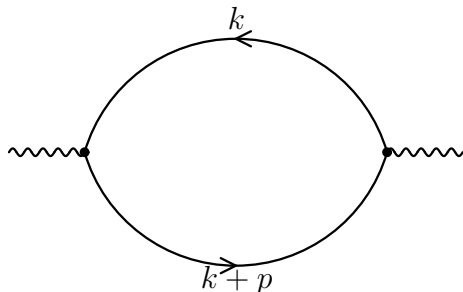
Quark contribution



Fermion loop -1

$$i(p^2 g_{\mu\nu} - p_\mu p_\nu) \Pi(p^2) =$$
$$- T_F n_f \int \frac{d^d k}{(2\pi)^d} \text{Tr} i g_0 \gamma_\mu i \frac{\not{k} + \not{p}}{(k+p)^2} i g_0 \gamma_\nu \frac{\not{k}}{k^2}$$

Quark contribution



Fermion loop -1

$$i(p^2 g_{\mu\nu} - p_\mu p_\nu) \Pi(p^2) = -T_F n_f \int \frac{d^d k}{(2\pi)^d} \text{Tr} i g_0 \gamma_\mu i \frac{\not{k} + \not{p}}{(k+p)^2} i g_0 \gamma_\nu \frac{\not{k}}{k^2}$$

In d dimensions $\delta_\mu^\mu = d$

$$\Pi(p^2) = \frac{-iT_F n_f g_0^2}{(d-1)(-p^2)} \int \frac{d^d k}{(2\pi)^d} \frac{\text{Tr} \gamma_\mu (\not{k} + \not{p}) \gamma^\mu \not{k}}{[-(k+p)^2] (-k^2)}$$

γ matrices in d dimensions

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

γ matrices in d dimensions

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

$$\gamma_\mu \gamma^\mu = d$$

γ matrices in d dimensions

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

$$\gamma_\mu \gamma^\mu = d$$

$$\gamma_\mu \not{a} \gamma^\mu = \gamma_\mu (-\gamma^\mu \not{a} + 2a^\mu) = -(d-2)\not{a}$$

γ matrices in d dimensions

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

$$\gamma_\mu \gamma^\mu = d$$

$$\gamma_\mu \not{a} \gamma^\mu = \gamma_\mu (-\gamma^\mu \not{a} + 2a^\mu) = -(d-2)\not{a}$$

$$\begin{aligned} \gamma_\mu \not{a} \not{b} \gamma^\mu &= \gamma_\mu \not{a} (-\gamma^\mu \not{b} + 2b^\mu) = (d-2)\not{a} \not{b} + 2\not{b} \not{a} \\ &= 4a \cdot b + (d-4)\not{a} \not{b} \end{aligned}$$

γ matrices in d dimensions

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

$$\gamma_\mu \gamma^\mu = d$$

$$\gamma_\mu \not{a} \gamma^\mu = \gamma_\mu (-\gamma^\mu \not{a} + 2a^\mu) = -(d-2)\not{a}$$

$$\begin{aligned}\gamma_\mu \not{a} \not{b} \gamma^\mu &= \gamma_\mu \not{a} (-\gamma^\mu \not{b} + 2b^\mu) = (d-2)\not{a} \not{b} + 2\not{b} \not{a} \\ &= 4a \cdot b + (d-4)\not{a} \not{b}\end{aligned}$$

$$\begin{aligned}\gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu &= \gamma_\mu \not{a} \not{b} (-\gamma^\mu \not{c} + 2c^\mu) = -4a \cdot b - (d-4)\not{a} \not{b} \not{c} + 2\not{c} \not{a} \not{b} \\ &= -2\not{c} \not{b} \not{a} - (d-4)\not{a} \not{b} \not{c}\end{aligned}$$

γ matrices in d dimensions

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$$

$$\gamma_\mu \gamma^\mu = d$$

$$\gamma_\mu \not{a} \gamma^\mu = \gamma_\mu (-\gamma^\mu \not{a} + 2a^\mu) = -(d-2)\not{a}$$

$$\begin{aligned}\gamma_\mu \not{a} \not{b} \gamma^\mu &= \gamma_\mu \not{a} (-\gamma^\mu \not{b} + 2b^\mu) = (d-2)\not{a} \not{b} + 2\not{b} \not{a} \\ &= 4a \cdot b + (d-4)\not{a} \not{b}\end{aligned}$$

$$\begin{aligned}\gamma_\mu \not{a} \not{b} \not{c} \gamma^\mu &= \gamma_\mu \not{a} \not{b} (-\gamma^\mu \not{c} + 2c^\mu) = -4a \cdot b - (d-4)\not{a} \not{b} \not{c} + 2\not{c} \not{a} \not{b} \\ &= -2\not{c} \not{b} \not{a} - (d-4)\not{a} \not{b} \not{c}\end{aligned}$$

Convention: $\text{Tr } 1 = 4$

Gluon self-energy – quark contribution

$$\Pi(p^2) = \frac{d-2}{d-1} \frac{iT_F n_f g_0^2}{-p^2} \int \frac{d^d k}{(2\pi)^d} \frac{4(k+p) \cdot k}{[-(k+p)^2](-k^2)}$$

Gluon self-energy – quark contribution

$$\Pi(p^2) = \frac{d-2}{d-1} \frac{iT_F n_f g_0^2}{-p^2} \int \frac{d^d k}{(2\pi)^d} \frac{4(k+p) \cdot k}{[-(k+p)^2](-k^2)}$$

Set $-p^2 = 1$

$$D_1 = -(k+p)^2 \quad D_2 = -k^2$$

Gluon self-energy – quark contribution

$$\Pi(p^2) = \frac{d-2}{d-1} \frac{iT_F n_f g_0^2}{-p^2} \int \frac{d^d k}{(2\pi)^d} \frac{4(k+p) \cdot k}{[-(k+p)^2](-k^2)}$$

Set $-p^2 = 1$

$$D_1 = -(k+p)^2 \quad D_2 = -k^2$$

Multiplication table

$$p^2 = -1 \quad k^2 = -D_2 \quad p \cdot k = \frac{1}{2}(1 + D_2 - D_1)$$

Gluon self-energy – quark contribution

$$\Pi(p^2) = \frac{d-2}{d-1} \frac{iT_F n_f g_0^2}{-p^2} \int \frac{d^d k}{(2\pi)^d} \frac{4(k+p) \cdot k}{[-(k+p)^2](-k^2)}$$

Set $-p^2 = 1$

$$D_1 = -(k+p)^2 \quad D_2 = -k^2$$

Multiplication table

$$p^2 = -1 \quad k^2 = -D_2 \quad p \cdot k = \frac{1}{2}(1 + D_2 - D_1)$$

$$\Pi(p^2) = 2 \frac{d-2}{d-1} iT_F n_f e_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{-2D_2 + 1 + D_2 - D_1}{D_1 D_2}$$

Gluon self-energy – quark contribution

Restoring $-p^2$ by dimensionality

$$\Pi(p^2) = -T_F n_f \frac{g_0^2 (-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} 2 \frac{d-2}{d-1} G_1$$

Gluon self-energy – quark contribution

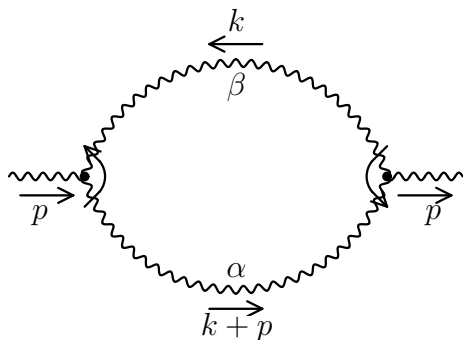
Restoring $-p^2$ by dimensionality

$$\Pi(p^2) = -T_F n_f \frac{g_0^2 (-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} 2 \frac{d-2}{d-1} G_1$$

Recalling G_1

$$\Pi(p^2) = T_F n_f \frac{g_0^2 (-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} 4 \frac{d-2}{(d-1)(d-3)(d-4)} g_1$$

Gluon contribution



Symmetry factor $\frac{1}{2}$. Colour factor C_A . Feynman gauge
 $a_0 = 1$

$$\Pi_1^\mu = -i \frac{1}{2} C_A g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{N}{k^2 (k+p)^2}$$

$$N = V_{\mu\alpha\beta}(p, -k-p, k) V^{\mu\beta\alpha}(-p, -k, k+p)$$

Gluon contribution

$$V_{\mu\alpha\beta}(p, -k-p, k) = (2k+p)_\mu g_{\alpha\beta} - (k-p)_\alpha g_{\beta\mu} - (k+2p)_\beta g_{\mu\alpha}$$

$$V^{\mu\beta\alpha}(-p, -k, k+p) \text{ — the same}$$

$$N = d [(2k+p)^2 + (k-p)^2 + (k+2p)^2] \\ - 2(2k+p) \cdot (k-p) - 2(2k+p) \cdot (k+2p) + 2(k+2p) \cdot (k-p)$$

Gluon contribution

$$V_{\mu\alpha\beta}(p, -k-p, k) = (2k+p)_\mu g_{\alpha\beta} - (k-p)_\alpha g_{\beta\mu} - (k+2p)_\beta g_{\mu\alpha}$$

$$V^{\mu\beta\alpha}(-p, -k, k+p) \text{ — the same}$$

$$N = d [(2k+p)^2 + (k-p)^2 + (k+2p)^2] \\ - 2(2k+p) \cdot (k-p) - 2(2k+p) \cdot (k+2p) + 2(k+2p) \cdot (k-p)$$

$$p^2 = -1 \quad k^2 = -D_2 \Rightarrow 0 \quad p \cdot k = \frac{1}{2}(1 - D_1 + D_2) \Rightarrow \frac{1}{2}$$

$$N \Rightarrow -3(d-1)$$

Gluon contribution

$$V_{\mu\alpha\beta}(p, -k-p, k) = (2k+p)_\mu g_{\alpha\beta} - (k-p)_\alpha g_{\beta\mu} - (k+2p)_\beta g_{\mu\alpha}$$

$$V^{\mu\beta\alpha}(-p, -k, k+p) \text{ — the same}$$

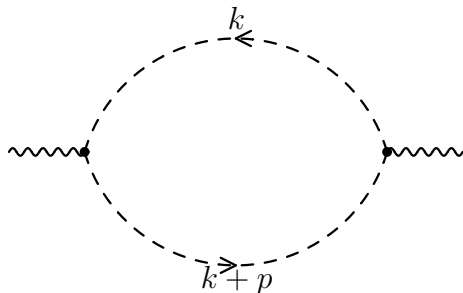
$$N = d \left[(2k+p)^2 + (k-p)^2 + (k+2p)^2 \right] \\ - 2(2k+p) \cdot (k-p) - 2(2k+p) \cdot (k+2p) + 2(k+2p) \cdot (k-p)$$

$$p^2 = -1 \quad k^2 = -D_2 \Rightarrow 0 \quad p \cdot k = \frac{1}{2}(1 - D_1 + D_2) \Rightarrow \frac{1}{2}$$

$$N \Rightarrow -3(d-1)$$

$$\Pi_{1\mu}^{\mu} = -\frac{3}{2} C_A \frac{g_0^2 (-p^2)^{1-\varepsilon}}{(4\pi)^{d/2}} G_1(d-1)$$

Ghost contribution



Fermion loop gives -1 . Colour factor C_A

$$\Pi_{2\mu}^{\mu} = iC_A g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{k \cdot (k+p)}{k^2 (k+p)^2} = -\frac{1}{2} C_A \frac{g_0^2 (-p^2)^{1-\epsilon}}{(4\pi)^{d/2}} G_1$$

Result

$$\Pi_g(p^2) = -\frac{\Pi_{1\mu}^\mu + \Pi_{2\mu}^\mu}{(d-1)(-p^2)} = C_A \frac{g_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} G_1 \frac{3d-2}{2(d-1)}$$

Result

$$\Pi_g(p^2) = -\frac{\Pi_{1\mu}^\mu + \Pi_{2\mu}^\mu}{(d-1)(-p^2)} = C_A \frac{g_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} G_1 \frac{3d-2}{2(d-1)}$$

In an arbitrary covariant gauge $\xi = 1 - a_0$

$$\begin{aligned} \Pi_g(p^2) &= C_A \frac{g_0^2(-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} \frac{G_1}{2(d-1)} \\ &\times \left[3d-2 + (d-1)(2d-7)\xi - \frac{1}{4}(d-1)(d-4)\xi^2 \right] \end{aligned}$$

Gluon field renormalization

$$p^2 D_{\perp}(p^2) = 1 + \frac{\alpha_s(\mu)}{4\pi\varepsilon} e^{-L\varepsilon} \left[-\frac{1}{2} \left(a - \frac{13}{3} \right) C_A - \frac{4}{3} T_F n_f \right. \\ \left. + \left(\frac{9a^2 + 18a + 97}{36} C_A - \frac{20}{9} T_F n_f \right) \varepsilon \right]$$

$$L = \log \frac{-p^2}{\mu^2}$$

Gluon field renormalization

$$p^2 D_{\perp}(p^2) = 1 + \frac{\alpha_s(\mu)}{4\pi\varepsilon} e^{-L\varepsilon} \left[-\frac{1}{2} \left(a - \frac{13}{3} \right) C_A - \frac{4}{3} T_F n_f \right. \\ \left. + \left(\frac{9a^2 + 18a + 97}{36} C_A - \frac{20}{9} T_F n_f \right) \varepsilon \right]$$

$$L = \log \frac{-p^2}{\mu^2}$$

$$Z_A = 1 - \frac{\alpha_s}{4\pi\varepsilon} \left[\frac{1}{2} \left(a - \frac{13}{3} \right) C_A + \frac{4}{3} T_F n_f \right]$$

$$\gamma_A = \left[\left(a - \frac{13}{3} \right) C_A + \frac{8}{3} T_F n_f \right] \frac{\alpha_s}{4\pi} + \dots$$

Photon field renormalization

Transverse propagator

$$p^2 D_{\perp}(p^2) = \frac{1}{1 - \Pi(p^2)} = 1 + \frac{e_0^2 (-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} 4 \frac{d-2}{(d-1)(d-3)(d-4)} g_1$$

Photon field renormalization

Transverse propagator

$$p^2 D_{\perp}(p^2) = \frac{1}{1 - \Pi(p^2)} = 1 + \frac{e_0^2 (-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} 4 \frac{d-2}{(d-1)(d-3)(d-4)} g_1$$

Re-expressing via $\alpha(\mu)$

$$p^2 D_{\perp}(p^2) = 1 + \frac{\alpha(\mu)}{4\pi} e^{-L\varepsilon} e^{\gamma\varepsilon} g_1 4 \frac{d-2}{(d-1)(d-3)(d-4)}$$

$$L = \log \frac{-p^2}{\mu^2}$$

Photon field renormalization

For g_1 at $\varepsilon \rightarrow 0$, using

$$\Gamma(1 + \varepsilon) = \exp \left[-\gamma\varepsilon + \sum_{n=2}^{\infty} \frac{(-1)^n \zeta_n}{n} \varepsilon^n \right]$$

$$\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n} \quad \zeta_2 = \frac{\pi^2}{6} \quad \zeta_3 \approx 1.202 \quad \zeta_4 = \frac{\pi^4}{90} \quad \dots$$

we have $e^{\gamma\varepsilon} g_1 = 1 + \mathcal{O}(\varepsilon^2)$

Photon field renormalization

For g_1 at $\varepsilon \rightarrow 0$, using

$$\Gamma(1 + \varepsilon) = \exp \left[-\gamma\varepsilon + \sum_{n=2}^{\infty} \frac{(-1)^n \zeta_n}{n} \varepsilon^n \right]$$

$$\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n} \quad \zeta_2 = \frac{\pi^2}{6} \quad \zeta_3 \approx 1.202 \quad \zeta_4 = \frac{\pi^4}{90} \quad \dots$$

we have $e^{\gamma\varepsilon} g_1 = 1 + \mathcal{O}(\varepsilon^2)$

$$p^2 D_{\perp}(p^2) = 1 - \frac{4}{3} \frac{\alpha(\mu)}{4\pi\varepsilon} \left[1 - \left(L - \frac{5}{3} \right) \varepsilon + \dots \right]$$

Photon field renormalization

For g_1 at $\varepsilon \rightarrow 0$, using

$$\Gamma(1 + \varepsilon) = \exp \left[-\gamma\varepsilon + \sum_{n=2}^{\infty} \frac{(-1)^n \zeta_n}{n} \varepsilon^n \right]$$

$$\zeta_n = \sum_{k=1}^{\infty} \frac{1}{k^n} \quad \zeta_2 = \frac{\pi^2}{6} \quad \zeta_3 \approx 1.202 \quad \zeta_4 = \frac{\pi^4}{90} \quad \dots$$

we have $e^{\gamma\varepsilon} g_1 = 1 + \mathcal{O}(\varepsilon^2)$

$$p^2 D_{\perp}(p^2) = 1 - \frac{4}{3} \frac{\alpha(\mu)}{4\pi\varepsilon} \left[1 - \left(L - \frac{5}{3} \right) \varepsilon + \dots \right]$$

This should be $Z_A(\alpha(\mu)) p^2 D_{\perp}^r(p^2; \mu)$:

$$Z_A(\alpha) = 1 - \frac{4}{3} \frac{\alpha}{4\pi\varepsilon}$$

$$p^2 D_{\perp}^r(p^2; \mu) = 1 + \frac{4}{3} \frac{\alpha(\mu)}{4\pi} \left(L - \frac{5}{3} \right)$$

RG equation

$D_{\perp}(p^2) = Z_A(\alpha(\mu))D_{\perp}^r(p^2; \mu)$ does not depend on μ :

$$\frac{\partial D_{\perp}^r(p^2; \mu)}{\partial \log \mu} + \gamma_A(\alpha(\mu))D_{\perp}^r(p^2; \mu) = 0$$

$$\gamma_A(\alpha(\mu)) = \frac{d \log Z_A(\alpha(\mu))}{d \log \mu}$$

RG equation

$D_{\perp}(p^2) = Z_A(\alpha(\mu))D_{\perp}^r(p^2; \mu)$ does not depend on μ :

$$\frac{\partial D_{\perp}^r(p^2; \mu)}{\partial \log \mu} + \gamma_A(\alpha(\mu))D_{\perp}^r(p^2; \mu) = 0$$

$$\gamma_A(\alpha(\mu)) = \frac{d \log Z_A(\alpha(\mu))}{d \log \mu}$$

For a **minimal renormalization constant**, using

$$\frac{d \log \alpha(\mu)}{d \log \mu} = -2\varepsilon + \dots$$

we obtain

$$\gamma(\alpha) = \gamma_0 \frac{\alpha}{4\pi} + \dots = -2z_1 \frac{\alpha}{4\pi} + \dots$$

$$Z(\alpha) = 1 - \frac{\gamma_0}{2} \frac{\alpha}{4\pi\varepsilon} + \dots$$

Anomalous dimension

$$\gamma_A(\alpha) = \frac{8}{3} \frac{\alpha}{4\pi}$$

Anomalous dimension

$$\gamma_A(\alpha) = \frac{8}{3} \frac{\alpha}{4\pi}$$

RG equation

$$\frac{\partial p^2 D_{\perp}^r}{\partial L} = \frac{\gamma_A}{2} p^2 D_{\perp}^r$$

with the initial condition

$$p^2 D_{\perp}^r(p^2; \mu^2 = -p^2) = 1 - \frac{20}{9} \frac{\alpha(\mu)}{4\pi}$$

is enough to reconstruct [the result](#)

Anomalous dimension

$$\gamma_A(\alpha) = \frac{8}{3} \frac{\alpha}{4\pi}$$

RG equation

$$\frac{\partial p^2 D_{\perp}^r}{\partial L} = \frac{\gamma_A}{2} p^2 D_{\perp}^r$$

with the initial condition

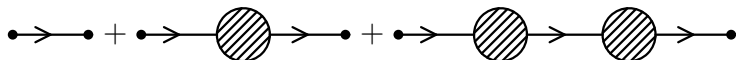
$$p^2 D_{\perp}^r(p^2; \mu^2 = -p^2) = 1 - \frac{20}{9} \frac{\alpha(\mu)}{4\pi}$$

is enough to reconstruct [the result](#)

$a_0 = Z_A(\alpha(\mu))a(\mu)$ does not depend on μ :

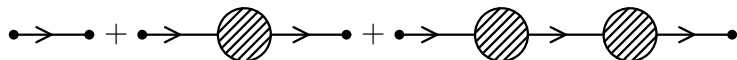
$$\frac{da(\mu)}{d \log \mu} + \gamma_A(\alpha(\mu))a(\mu) = 0$$

Quark propagator



$$iS(p) = iS_0(p) + iS_0(p)(-i)\Sigma(p)iS_0(p) \\ + iS_0(p)(-i)\Sigma(p)iS_0(p)(-i)\Sigma(p)iS_0(p) + \dots$$

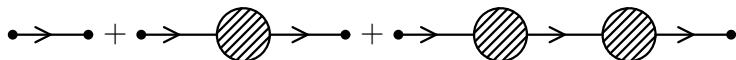
Quark propagator



$$iS(p) = iS_0(p) + iS_0(p)(-i)\Sigma(p)iS_0(p) \\ + iS_0(p)(-i)\Sigma(p)iS_0(p)(-i)\Sigma(p)iS_0(p) + \dots$$

$$S(p) = S_0(p) + S_0(p)\Sigma(p)S(p)$$

Quark propagator

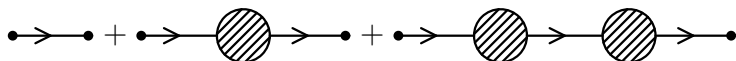


$$iS(p) = iS_0(p) + iS_0(p)(-i)\Sigma(p)iS_0(p) \\ + iS_0(p)(-i)\Sigma(p)iS_0(p)(-i)\Sigma(p)iS_0(p) + \dots$$

$$S(p) = S_0(p) + S_0(p)\Sigma(p)S(p)$$

$$S(p) = \frac{1}{S_0^{-1}(p) - \Sigma(p)}$$

Quark propagator



$$iS(p) = iS_0(p) + iS_0(p)(-i)\Sigma(p)iS_0(p) \\ + iS_0(p)(-i)\Sigma(p)iS_0(p)(-i)\Sigma(p)iS_0(p) + \dots$$

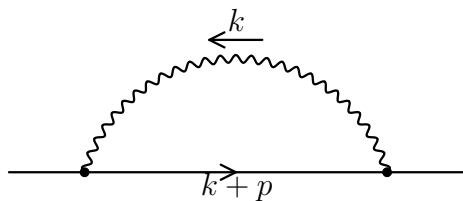
$$S(p) = S_0(p) + S_0(p)\Sigma(p)S(p)$$

$$S(p) = \frac{1}{S_0^{-1}(p) - \Sigma(p)}$$

Massless case: $\Sigma(p) = \not{p}\Sigma_V(p^2)$ (helicity conservation)

$$S(p) = \frac{1}{1 - \Sigma_V(p^2)} \frac{1}{\not{p}}$$

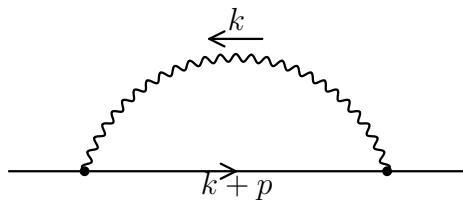
Quark self-energy



$$-i\not{p}\Sigma_V(p^2) = C_F \int \frac{d^d k}{(2\pi)^d} i g_0 \gamma^\mu i \frac{\not{k} + \not{p}}{(k+p)^2} i g_0 \gamma^\nu \frac{-i}{k^2} \left(g_{\mu\nu} - \xi \frac{k_\mu k_\nu}{k^2} \right)$$

where $\xi = 1 - a_0$.

Quark self-energy



$$-i\not{p}\Sigma_V(p^2) = C_F \int \frac{d^d k}{(2\pi)^d} i g_0 \gamma^\mu i \frac{\not{k} + \not{p}}{(k+p)^2} i g_0 \gamma^\nu \frac{-i}{k^2} \left(g_{\mu\nu} - \xi \frac{k_\mu k_\nu}{k^2} \right)$$

where $\xi = 1 - a_0$. Taking $\frac{1}{4} \text{Tr } \not{p}$

$$\Sigma_V(p^2) = \frac{i C_F g_0^2}{-p^2} \int \frac{d^d k}{(2\pi)^d} \frac{N}{D_1 D_2}$$

$$N = \frac{1}{4} \text{Tr } \not{p} \gamma^\mu (\not{k} + \not{p}) \gamma^\nu \left(g_{\mu\nu} + \xi \frac{k_\mu k_\nu}{D_2} \right)$$

Quark self-energy

Using multiplication table

$$\begin{aligned} N &= \frac{1}{4} \text{Tr } \not{p} \gamma_\mu (\not{k} + \not{p}) \gamma^\mu + \frac{\xi}{D_2} \frac{1}{4} \text{Tr } \not{p} \not{k} (\not{k} + \not{p}) \not{k} \\ &= -(d-2)(p^2 + p \cdot k) + \frac{\xi}{D_2} [k^2 p \cdot k + 2(p \cdot k)^2 - p^2 k^2] \\ &= \frac{1}{2} \left[d - 2 + \xi \left(\frac{1}{D_2} - 1 \right) \right] \end{aligned}$$

Quark self-energy

Using multiplication table

$$\begin{aligned} N &= \frac{1}{4} \text{Tr } \not{p} \gamma_\mu (\not{k} + \not{p}) \gamma^\mu + \frac{\xi}{D_2} \frac{1}{4} \text{Tr } \not{p} \not{k} (\not{k} + \not{p}) \not{k} \\ &= -(d-2)(p^2 + p \cdot k) + \frac{\xi}{D_2} [k^2 p \cdot k + 2(p \cdot k)^2 - p^2 k^2] \\ &= \frac{1}{2} \left[d - 2 + \xi \left(\frac{1}{D_2} - 1 \right) \right] \end{aligned}$$

$$\Sigma_V(p^2) = -C_F \frac{g_0^2 (-p^2)^{-\epsilon}}{(4\pi)^{d/2}} \frac{1}{2} [(d-2-\xi)G(1,1) + \xi G(1,2)]$$

Quark self-energy

$$\frac{G(n_1, n_2 + 1)}{G(n_1, n_2)} = - \frac{(d - 2n_1 - 2n_2)(d - n_1 - n_2 - 1)}{n_2(d - 2n_2 - 2)}$$

Quark self-energy

$$\frac{G(n_1, n_2 + 1)}{G(n_1, n_2)} = - \frac{(d - 2n_1 - 2n_2)(d - n_1 - n_2 - 1)}{n_2(d - 2n_2 - 2)}$$

$$\frac{G(1, 2)}{G(1, 1)} = -(d - 3)$$

Quark self-energy

$$\frac{G(n_1, n_2 + 1)}{G(n_1, n_2)} = - \frac{(d - 2n_1 - 2n_2)(d - n_1 - n_2 - 1)}{n_2(d - 2n_2 - 2)}$$

$$\frac{G(1, 2)}{G(1, 1)} = -(d - 3)$$

$$\Sigma_V(p^2) = -C_F \frac{g_0^2 (-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} \frac{d - 2}{2} a_0 G_1$$

Quark self-energy

$$\frac{G(n_1, n_2 + 1)}{G(n_1, n_2)} = - \frac{(d - 2n_1 - 2n_2)(d - n_1 - n_2 - 1)}{n_2(d - 2n_2 - 2)}$$

$$\frac{G(1, 2)}{G(1, 1)} = -(d - 3)$$

$$\Sigma_V(p^2) = -C_F \frac{g_0^2 (-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} \frac{d - 2}{2} a_0 G_1$$

$$\Sigma_V(p^2) = C_F \frac{g_0^2 (-p^2)^{-\varepsilon}}{(4\pi)^{d/2}} \frac{d - 2}{(d - 3)(d - 4)} a_0 g_1$$

Quark field renormalization

$$\not{p}S(p) = \frac{1}{1 - \Sigma_V(p^2)}$$

expressed via the **renormalized quantities**

$$\begin{aligned}\not{p}S(p) &= 1 + C_F \frac{\alpha_s(\mu)}{4\pi} e^{-L\varepsilon} e^{\gamma\varepsilon} g_1 a(\mu) \frac{d-2}{(d-3)(d-4)} \\ &= 1 - \frac{\alpha(\mu)}{4\pi\varepsilon} a(\mu) e^{-L\varepsilon} (1 + \varepsilon + \dots)\end{aligned}$$

Quark field renormalization

$$\not{p}S(p) = \frac{1}{1 - \Sigma_V(p^2)}$$

expressed via the **renormalized quantities**

$$\begin{aligned}\not{p}S(p) &= 1 + C_F \frac{\alpha_s(\mu)}{4\pi} e^{-L\varepsilon} e^{\gamma\varepsilon} g_1 a(\mu) \frac{d-2}{(d-3)(d-4)} \\ &= 1 - \frac{\alpha(\mu)}{4\pi\varepsilon} a(\mu) e^{-L\varepsilon} (1 + \varepsilon + \dots)\end{aligned}$$

should be $Z_q(\alpha(\mu), a(\mu))\not{p}S_r(p; \mu)$:

$$Z_q(\alpha, a) = 1 - C_F a \frac{\alpha_s}{4\pi\varepsilon}$$

$$\not{p}S_r(p) = 1 + C_F a(\mu) (L - 1) \frac{\alpha_s(\mu)}{4\pi}$$

Quark field renormalization

$$\not{p}S(p) = \frac{1}{1 - \Sigma_V(p^2)}$$

expressed via the **renormalized quantities**

$$\begin{aligned}\not{p}S(p) &= 1 + C_F \frac{\alpha_s(\mu)}{4\pi} e^{-L\varepsilon} e^{\gamma\varepsilon} g_1 a(\mu) \frac{d-2}{(d-3)(d-4)} \\ &= 1 - \frac{\alpha(\mu)}{4\pi\varepsilon} a(\mu) e^{-L\varepsilon} (1 + \varepsilon + \dots)\end{aligned}$$

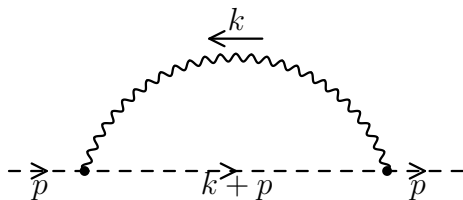
should be $Z_q(\alpha(\mu), a(\mu))\not{p}S_r(p; \mu)$:

$$Z_q(\alpha, a) = 1 - C_F a \frac{\alpha_s}{4\pi\varepsilon}$$

$$\not{p}S_r(p) = 1 + C_F a(\mu) (L - 1) \frac{\alpha_s(\mu)}{4\pi}$$

$$\gamma_q(\alpha_s, a) = 2a \frac{\alpha_s}{4\pi}$$

Ghost self-energy



$$\begin{aligned}\Sigma(p^2) &= -iC_A g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{p^\mu (k+p)^\nu}{k^2 (k+p)^2} \left(g_{\mu\nu} - \xi \frac{k_\mu k_\nu}{k^2} \right) \\ &= C_A \frac{g_0^2 (-p^2)^{1-\varepsilon}}{(4\pi)^{d/2}} \left[-\frac{1}{2} G(1, 1) + \frac{\xi}{4} G(1, 2) \right] \\ &= -\frac{1}{4} C_A \frac{g_0^2 (-p^2)^{1-\varepsilon}}{(4\pi)^{d/2}} G_1 [d-1 - (d-3)a_0]\end{aligned}$$

Ghost field renormalization

$$G(p) = \frac{1}{p^2 - \Sigma(p^2)} = \frac{1}{p^2} \left[1 + C_A \frac{\alpha_s(\mu)}{4\pi\varepsilon} e^{-L\varepsilon} \frac{3 - a + 4\varepsilon}{4} \right]$$

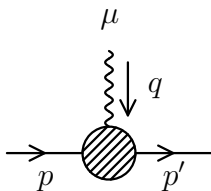
Ghost field renormalization

$$G(p) = \frac{1}{p^2 - \Sigma(p^2)} = \frac{1}{p^2} \left[1 + C_A \frac{\alpha_s(\mu)}{4\pi\varepsilon} e^{-L\varepsilon} \frac{3 - a + 4\varepsilon}{4} \right]$$

$$Z_c = 1 + C_A \frac{3 - a}{4} \frac{\alpha_s}{4\pi\varepsilon}$$

$$\gamma_c = -C_A \frac{3 - a}{2} \frac{\alpha_s}{4\pi} + \dots$$

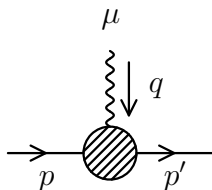
Vertex



$$= ig_0 t^a \Gamma^\mu(p, p')$$

$$\Gamma^\mu(p, p') = \gamma^\mu + \Lambda^\mu(p, p')$$

Vertex



$$= ig_0 t^a \Gamma^\mu(p, p')$$

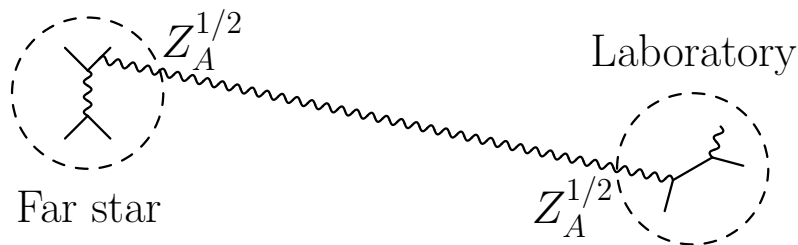
$$\Gamma^\mu(p, p') = \gamma^\mu + \Lambda^\mu(p, p')$$

When expressed via renormalized quantities,

$$\Gamma^\mu = Z_\Gamma \Gamma_r^\mu$$

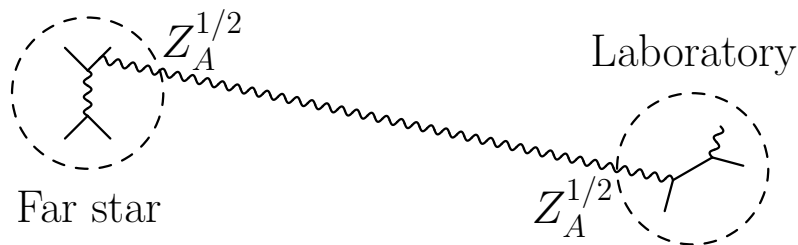
Matrix element

S -matrix element = vertex $\times Z_i^{1/2}$ for each i



Matrix element

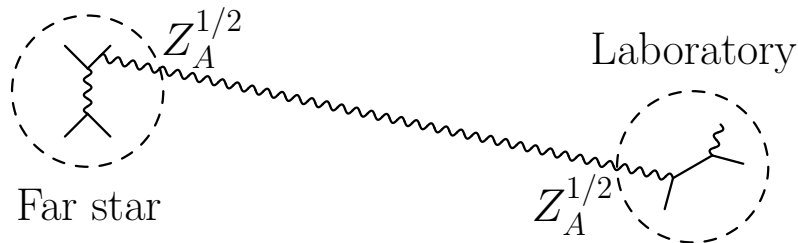
S -matrix element = vertex $\times Z_i^{1/2}$ for each i



The physical matrix element
 $e_0 \Gamma Z_\psi Z_A^{1/2} = e \Gamma_r Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2}$ must be finite

Matrix element

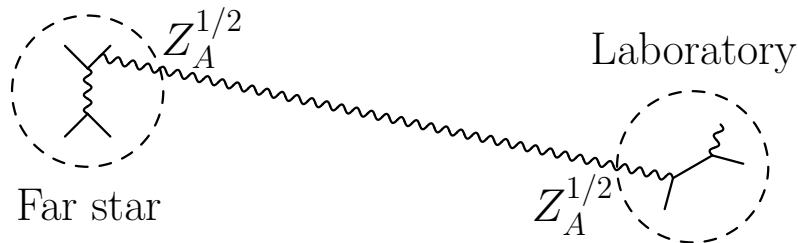
S -matrix element = vertex $\times Z_i^{1/2}$ for each i



The physical matrix element
 $e_0 \Gamma Z_\psi Z_A^{1/2} = e \Gamma_r Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2}$ must be finite
 $Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2}$ must be finite

Matrix element

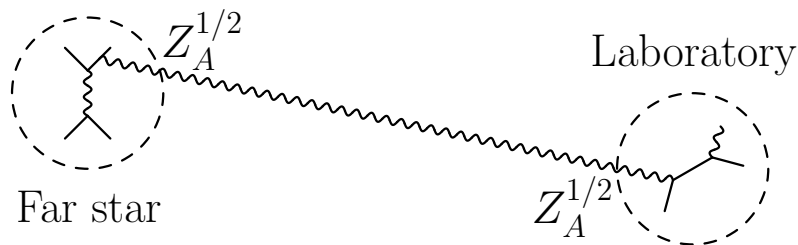
S -matrix element = vertex $\times Z_i^{1/2}$ for each i



The physical matrix element
 $e_0 \Gamma Z_\psi Z_A^{1/2} = e \Gamma_r Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2}$ must be finite
 $Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2}$ must be finite = 1

Matrix element

S -matrix element = vertex $\times Z_i^{1/2}$ for each i



The physical matrix element
 $e_0 \Gamma Z_\psi Z_A^{1/2} = e \Gamma_r Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2}$ must be finite
 $Z_\alpha^{1/2} Z_\Gamma Z_\psi Z_A^{1/2}$ must be finite = 1

$$Z_\alpha = (Z_\Gamma Z_\psi)^{-2} Z_A^{-1}$$

Quark–gluon vertex

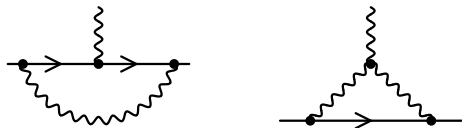
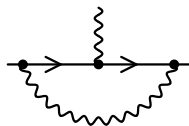


Diagram 1: like in QED, but with colour factor

$$\Lambda_1^\alpha = \left(C_F - \frac{C_A}{2} \right) \frac{\alpha_s}{4\pi\epsilon} \gamma^\alpha$$

Diagram 1

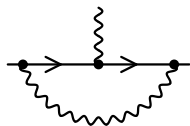
Direct calculation



$$ie_0\Lambda^\alpha = \int \frac{d^d k}{(2\pi)^d} ie_0\gamma^\mu i \frac{\not{k}}{k^2} ie_0\gamma^\alpha i \frac{\not{k}}{k^2} ie_0\gamma^\nu$$
$$\times \frac{-i}{k^2} \left(g_{\mu\nu} - \xi \frac{k_\mu k_\nu}{k^2} \right)$$

$$\Lambda^\alpha = -ie_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_\mu \not{k} \gamma^\alpha \not{k} \gamma^\mu - \xi k^2 \gamma^\alpha}{(k^2)^2}$$

Direct calculation



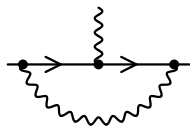
$$ie_0\Lambda^\alpha = \int \frac{d^d k}{(2\pi)^d} ie_0\gamma^\mu i \frac{\not{k}}{k^2} ie_0\gamma^\alpha i \frac{\not{k}}{k^2} ie_0\gamma^\nu$$
$$\times \frac{-i}{k^2} \left(g_{\mu\nu} - \xi \frac{k_\mu k_\nu}{k^2} \right)$$

$$\Lambda^\alpha = -ie_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_\mu \not{k} \gamma^\alpha \not{k} \gamma^\mu - \xi k^2 \gamma^\alpha}{(k^2)^2}$$

Averaging $\not{k} \gamma^\alpha \not{k} \rightarrow (k^2/d)\gamma_\nu \gamma^\alpha \gamma^\nu$

$$\Lambda^\alpha = -ie_0^2 a_0 \gamma^\alpha \int \frac{d^d k}{(2\pi)^d} \frac{1}{(-k^2)^2}$$

Direct calculation



$$ie_0\Lambda^\alpha = \int \frac{d^d k}{(2\pi)^d} ie_0\gamma^\mu i \frac{\not{k}}{k^2} ie_0\gamma^\alpha i \frac{\not{k}}{k^2} ie_0\gamma^\nu$$

$$\times \frac{-i}{k^2} \left(g_{\mu\nu} - \xi \frac{k_\mu k_\nu}{k^2} \right)$$

$$\Lambda^\alpha = -ie_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_\mu \not{k} \gamma^\alpha \not{k} \gamma^\mu - \xi k^2 \gamma^\alpha}{(k^2)^2}$$

Averaging $\not{k} \gamma^\alpha \not{k} \rightarrow (k^2/d)\gamma_\nu \gamma^\alpha \gamma^\nu$

$$\Lambda^\alpha = -ie_0^2 a_0 \gamma^\alpha \int \frac{d^d k}{(2\pi)^d} \frac{1}{(-k^2)^2}$$

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(-k^2)^2} \Big|_{UV} = \frac{i}{8\pi^2} \int_\lambda^\infty k^{-1-2\varepsilon} dk = \frac{i\lambda^{-2\varepsilon}}{(4\pi)^2 \varepsilon} = \frac{i}{(4\pi)^2} \frac{1}{\varepsilon}$$

Direct calculation

Any infrared regulator can be used

$$\begin{aligned} \int \frac{d^d k}{(2\pi)^d} \frac{1}{(-k^2)^2} \Big|_{UV} &= \int \frac{d^d k}{(2\pi)^d} \frac{1}{(m^2 - k^2)^2} \\ &= \frac{im^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) = \frac{i}{(4\pi)^2} \frac{1}{\varepsilon} \end{aligned}$$

Direct calculation

Any infrared regulator can be used

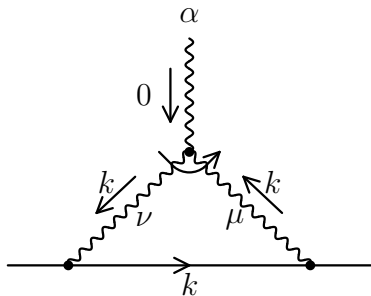
$$\begin{aligned}\int \frac{d^d k}{(2\pi)^d} \frac{1}{(-k^2)^2} \Big|_{UV} &= \int \frac{d^d k}{(2\pi)^d} \frac{1}{(m^2 - k^2)^2} \\ &= \frac{im^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) = \frac{i}{(4\pi)^2} \frac{1}{\varepsilon}\end{aligned}$$

$$\Gamma^\alpha = \gamma^\alpha \left[1 + a(\mu) \frac{\alpha(\mu)}{4\pi\varepsilon} \right]$$

$$Z_\Gamma = 1 + a \frac{\alpha}{4\pi\varepsilon}$$

agrees with Z_ψ

Diagram 2



Feynman gauge $a_0 = 1$
 Colour factor $C_A/2$

$$\Lambda_2^\alpha = i \frac{C_A}{2} g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{\gamma_\mu \not{k} \gamma_\nu}{(k^2)^3} V^{\alpha\nu\mu}(0, -k, k)$$

$$V^{\alpha\nu\mu}(0, -k, k) = 2k^\alpha g^{\mu\nu} - k^\mu g^{\nu\alpha} - k^\nu g^{\mu\alpha}$$

$$\Lambda_2^\alpha = i \frac{C_A}{2} g_0^2 \int \frac{d^d k}{(2\pi)^d} \frac{2\gamma_\mu \not{k} \gamma^\mu k^\alpha - 2k^2 \gamma^\alpha}{(k^2)^3}$$

Result

Averaging $\not{k}k^\alpha \rightarrow (k^2/d)\gamma^\alpha$

$$\Lambda_2^\alpha = \frac{3}{2}C_A \frac{\alpha_s}{4\pi\epsilon} \gamma^\alpha$$

Result

Averaging $k k^\alpha \rightarrow (k^2/d)\gamma^\alpha$

$$\Lambda_2^\alpha = \frac{3}{2} C_A \frac{\alpha_s}{4\pi\epsilon} \gamma^\alpha$$

Arbitrary covariant gauge

$$\Lambda_2^\alpha = \frac{3}{4} (1 + a) C_A \frac{\alpha_s}{4\pi\epsilon} \gamma^\alpha$$

Result

Averaging $\not{k}k^\alpha \rightarrow (k^2/d)\gamma^\alpha$

$$\Lambda_2^\alpha = \frac{3}{2}C_A \frac{\alpha_s}{4\pi\epsilon} \gamma^\alpha$$

Arbitrary covariant gauge

$$\Lambda_2^\alpha = \frac{3}{4}(1+a)C_A \frac{\alpha_s}{4\pi\epsilon} \gamma^\alpha$$

$$\Lambda^\alpha = \left(C_F a + C_A \frac{a+3}{4} \right) \frac{\alpha_s}{4\pi\epsilon} \gamma^\alpha$$

Result

Averaging $\not{k}k^\alpha \rightarrow (k^2/d)\gamma^\alpha$

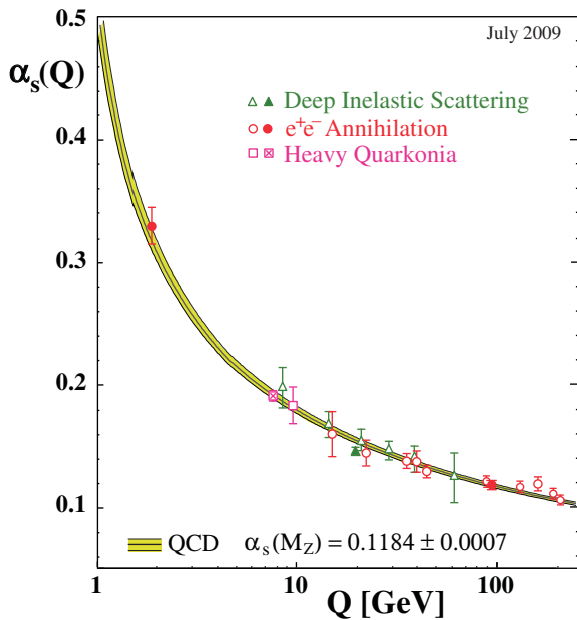
$$\Lambda_2^\alpha = \frac{3}{2}C_A \frac{\alpha_s}{4\pi\epsilon} \gamma^\alpha$$

Arbitrary covariant gauge

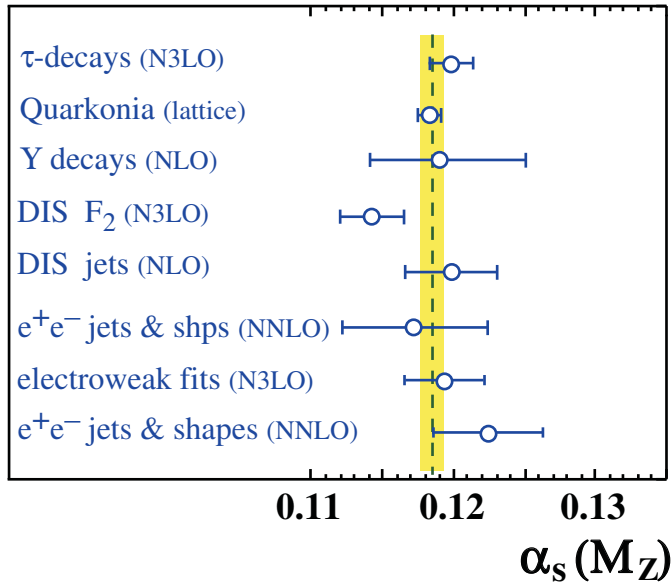
$$\Lambda_2^\alpha = \frac{3}{4}(1+a)C_A \frac{\alpha_s}{4\pi\epsilon} \gamma^\alpha$$

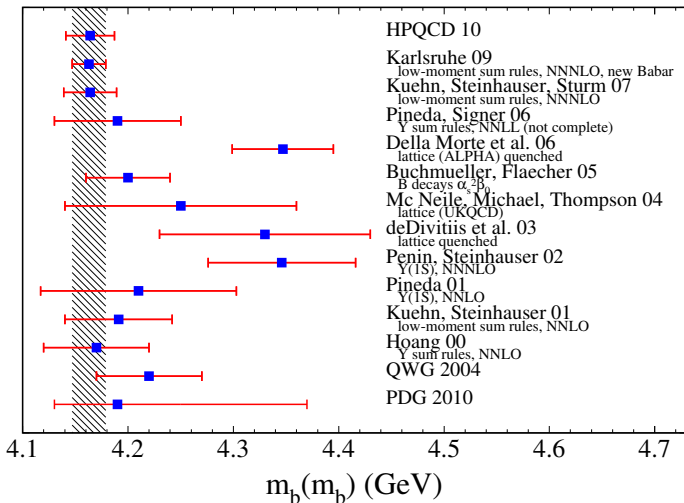
$$\Lambda^\alpha = \left(C_F a + C_A \frac{a+3}{4} \right) \frac{\alpha_s}{4\pi\epsilon} \gamma^\alpha$$

$$Z_\Gamma = 1 + \left(C_F a + C_A \frac{a+3}{4} \right) \frac{\alpha_s}{4\pi\epsilon}$$

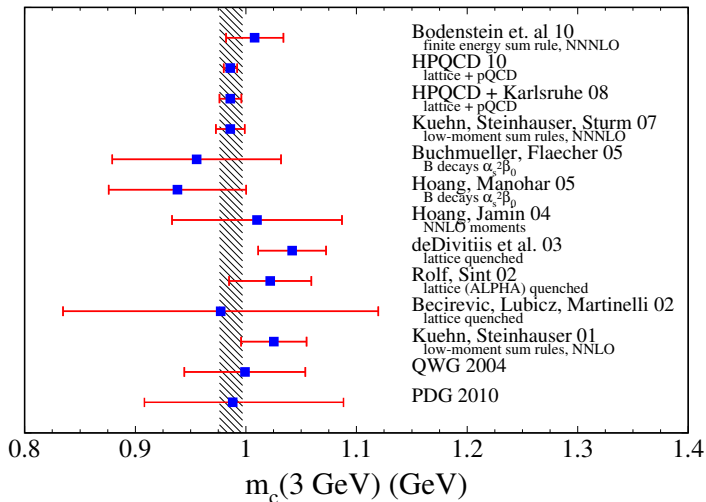
$\alpha_s(Q)$ 

$$\alpha_s(m_Z)$$

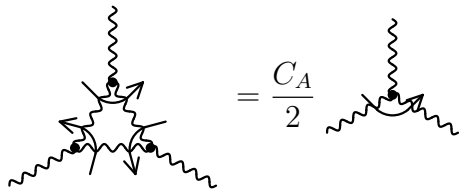


$m_b(m_b)$ 

$m_c(3 \text{ GeV})$



Example 3

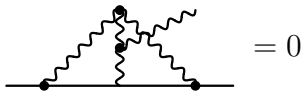


The diagram shows the reduction of a three-gluon vertex. On the left, a central vertex is connected to three external wavy gluon lines. Each of the two lower external lines has a small loop of a ghost line (represented by a dashed line with arrows) attached to it. This diagram is equated to a simpler diagram on the right, which is the same three-gluon vertex but without the ghost loops. The equality is indicated by an equals sign and the factor $\frac{C_A}{2}$.

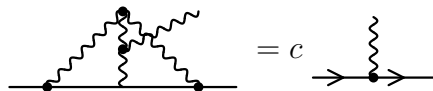
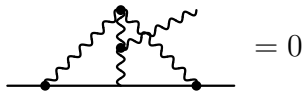
$$= \frac{C_A}{2}$$

$$i f^{adf} i f^{bed} i f^{cfe} = \frac{C_A}{2} i f^{abc}$$

Example 4



Example 4



Example 4

