Basics of QCD PT and APT

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Contents

- **QCD**: Quarks, hadrons and colour

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- **Renormalization**: General scheme and illustration

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- **Renormgroup (RG)**: **QED** and **QCD** β-functions at 1 loop approximation

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- **DGLAP Evolution** for inclusive reactions: Bjorken scaling in DIS and its violation in **QCD**

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- **Analytic PT**: Resummation of perturbations

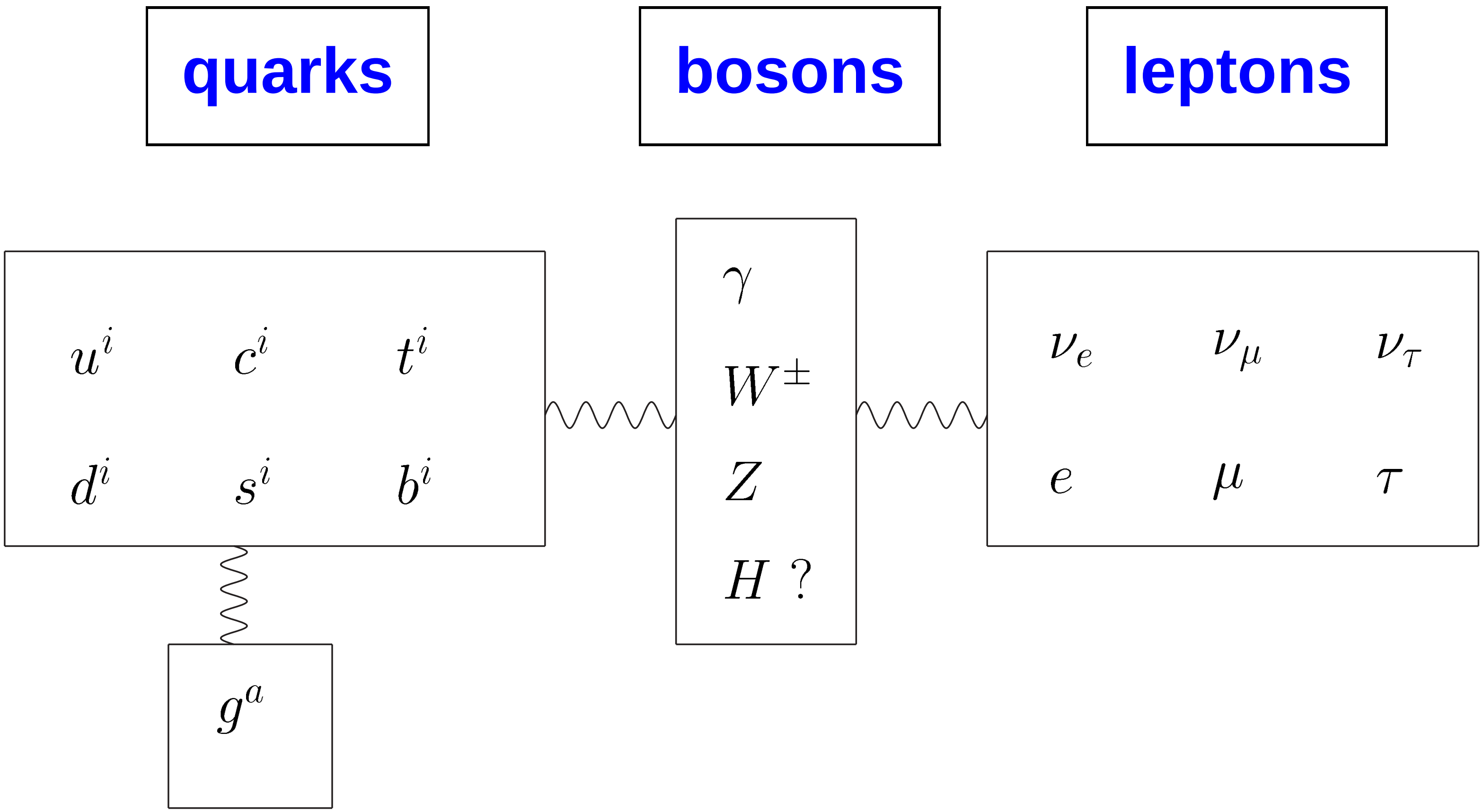
Quarks inside,

Hadrons outside!

That’s QCD!

Particles of Standard Model

Particles of **Standard Model**: $i = 1, 2, 3$ and $a = 1, \ldots 8$ — colour indices of quarks and gluons.



QCD $+$ **Electroweak physics**

Particles of Standard Model

quarks				
symbol	*name*	*el. charge* $[Q_e]$	*bar.charge*	*mass* $[m_p]$
$\boldsymbol{U}$	верхние (**up**)			
$\boldsymbol{u}$	up	$\mathbf{+2/3}$	$\mathbf{+1/3}$	$\mathbf{0.003}$
$\boldsymbol{c}$	charm	$\mathbf{+2/3}$	$\mathbf{+1/3}$	$\mathbf{1.3}$
$\boldsymbol{t}$	top	$\mathbf{+2/3}$	$\mathbf{+1/3}$	$\mathbf{180}$
$\boldsymbol{D}$	нижние (**down**)			
$\boldsymbol{d}$	down	$\mathbf{-1/3}$	$\mathbf{+1/3}$	$\mathbf{0.006}$
$\boldsymbol{s}$	strange	$\mathbf{-1/3}$	$\mathbf{+1/3}$	$\mathbf{0.13}$
$\boldsymbol{b}$	beauty	$\mathbf{-1/3}$	$\mathbf{+1/3}$	$\mathbf{4.5}$

QCD: Lagrangian, quarks and gluons

Gauge-invariant Lagrangian of QCD

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_{q=u,d,s,\ldots} \overline{\psi_q} (i\hat{D} - m_q) \psi_q$$

contains only gluon ($G^a_{\mu\nu}(x)$) and quark ($\psi_q(x)$) fields.

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contains only gluon ($G^a_{\mu\nu}(x)$) and quark ($\psi_q(x)$) fields.
These fields has color degrees of freedom: 3 for quarks $\psi^A_q(x)$ ($A = \mathbf{1}, 2, \mathbf{3}$) and 8 for gluons $G^a_{\mu\nu}(x)$ ($a = \mathbf{1}, \ldots, \mathbf{8}$).

QCD: Lagrangian, quarks and gluons

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Interaction is hidden in $G^{a}_{\mu\nu}$ and covariant derivative D^{AB}_{μ}:

$$\begin{aligned} G^{a}_{\mu\nu} &= \partial_\mu A^{a}_{\nu} - \partial_\nu A^{a}_{\mu} + g_s f^{abc} A^{b}_{\mu} A^{c}_{\nu} \\ D^{AB}_{\mu} &= \partial_\mu - i g_s \left(t^{a}\right)^{AB} A^{a}_{\mu} \end{aligned}$$

QCD: Lagrangian, quarks and gluons

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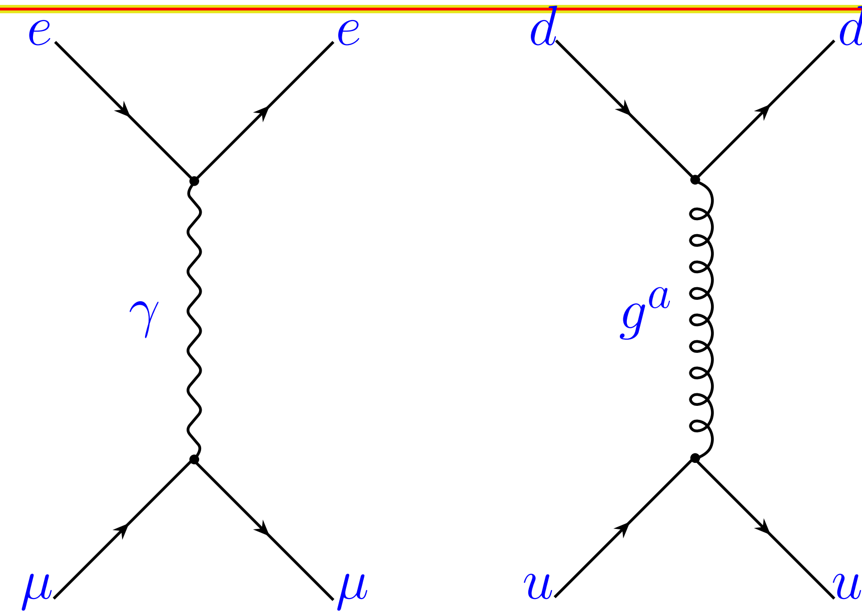
It is nonlinear due to **Non-Abelian** character ($f^{abc} \neq 0$).

QCD: Coloured gluons ⇒ Confinement

Consider $e\mu$- and qq-scattering (for d- and u-flavors):
wavy line denotes **photon** and curved line – **gluon**.



Comparison:

Parameter	Photon	Gluon
Mass	$\mathbf{0}$	$\mathbf{0}$
Spin	$\mathbf{1}$	$\mathbf{1}$
Vertex	$e\gamma_\mu$	$g_s\gamma_\mu(t^a)_{ij}$
Charge	$\mathbf{0}$	$\neq \mathbf{0}$

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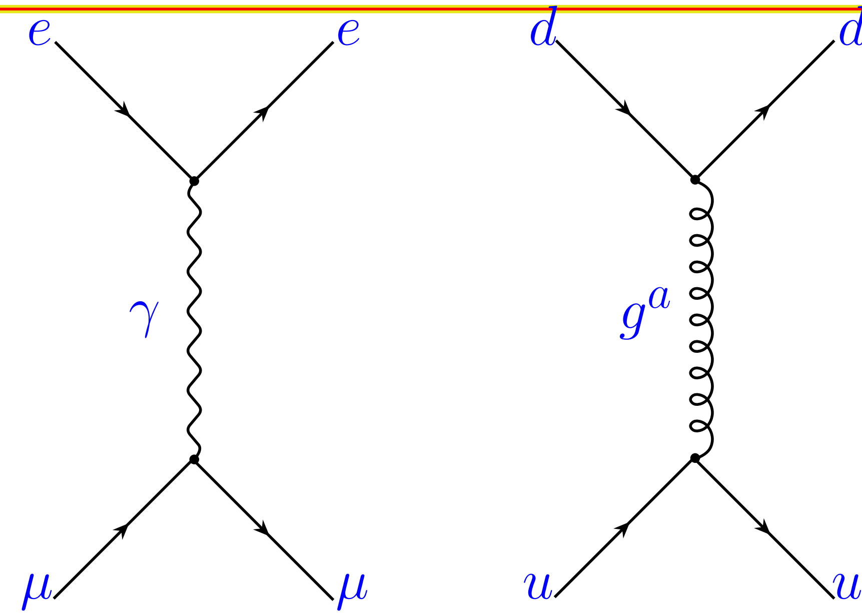

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Non-Abelian character of QCD $\Rightarrow$ charged **gluons**.

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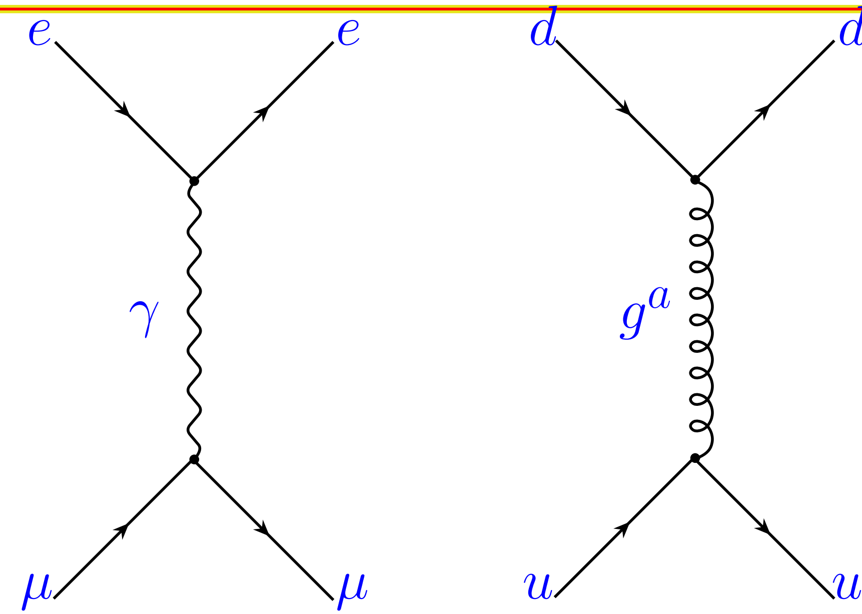

Comparison:

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Mass	0	0
Spin	1	1
Vertex	$e\gamma_\mu$	$g_s\gamma_\mu(t^a)_{ij}$
Charge	0	$\neq 0$

Coloured **gluons** $\Rightarrow$ **confinement**!

Massless QCD: What are Hadrons?

PS- and V-mesons composed of u- and d-quarks		
meson type	**PS**	**V**
composition	$\pi^0[\overline{u}u - \overline{d}d]$, $\pi^{\pm}[\overline{u}d, \overline{d}u]$	$\rho^0(\omega)[\overline{u}u - \overline{d}d]$, $\rho^{\pm}[\overline{u}d, \overline{d}u]$
mass	140 **MeV**	770(780) **MeV**

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mass	140 **MeV**	770(780) **MeV**

Baryons composed of u- and d-quarks			
composition	$p[uud]$	$n[udd]$	$\Delta^{++}[uuu]$, $\Delta^{+}[uud]$, $\Delta^{0}[udd]$, $\Delta^{-}[ddd]$
mass	938 **MeV**	939 **MeV**	**1232 MeV**

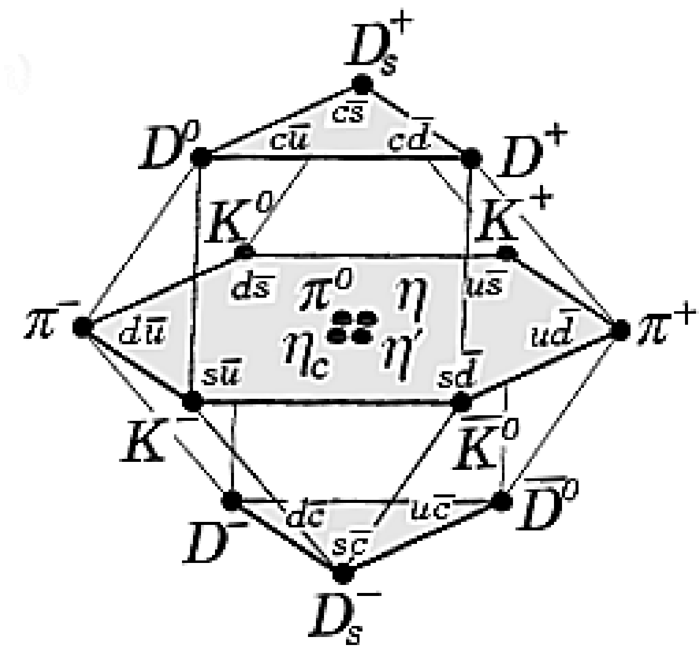
Hadrons of Standard Model

mesons $= q\bar{q}$, $qg\bar{q}$ или gg

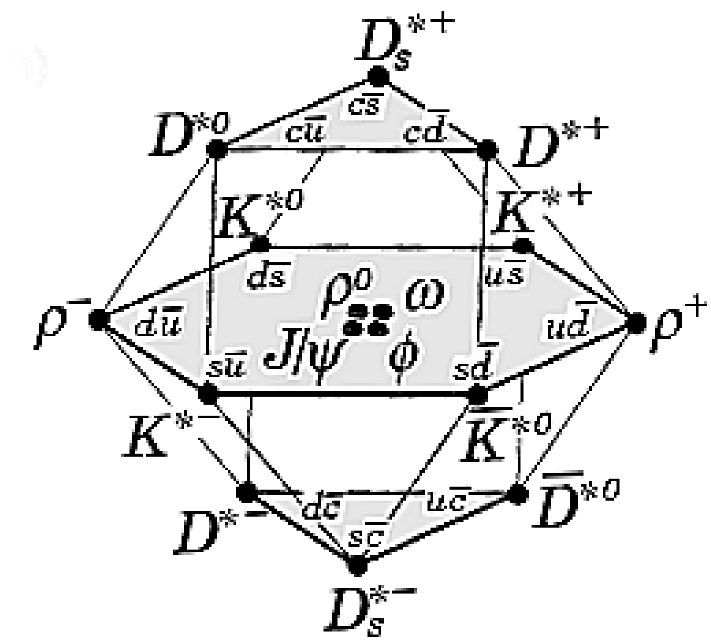
symbol	name	el. charge $[Q_e]$	mass $[m_p]$	spin
π	пион	$0, \pm 1$	0.15	0
K	каон	$0, \pm 1$	0.5	0
η	η-мезон	0	0.6	0
σ	σ-мезон	0	0.64	0
ρ	ρ-мезон	$0, \pm 1$	0.8	1
ω	ω-мезон	0	0.8	1
η'	η'-мезон	0	1	0

Hadrons of Standard Model

Mesonic 16-plets, build up from $\boldsymbol{u}$-, $\boldsymbol{d}$-, $\boldsymbol{s}$- and $\boldsymbol{c}$-quarks.



pseudoscalar mesons



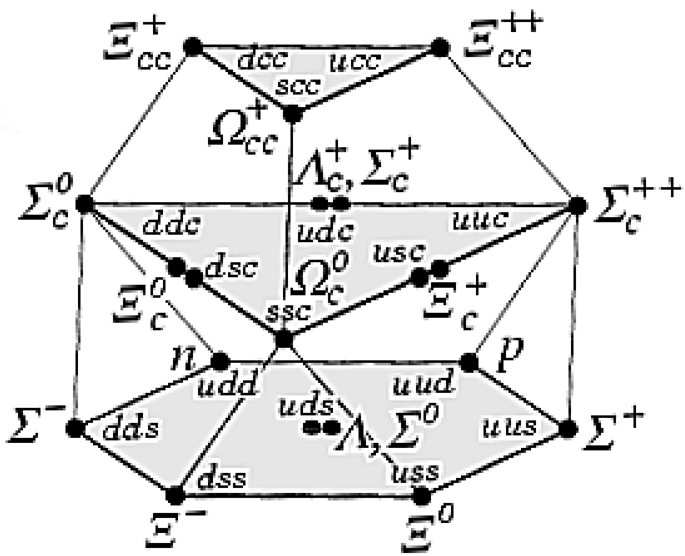
vector mesons

Hadrons of Standard Model

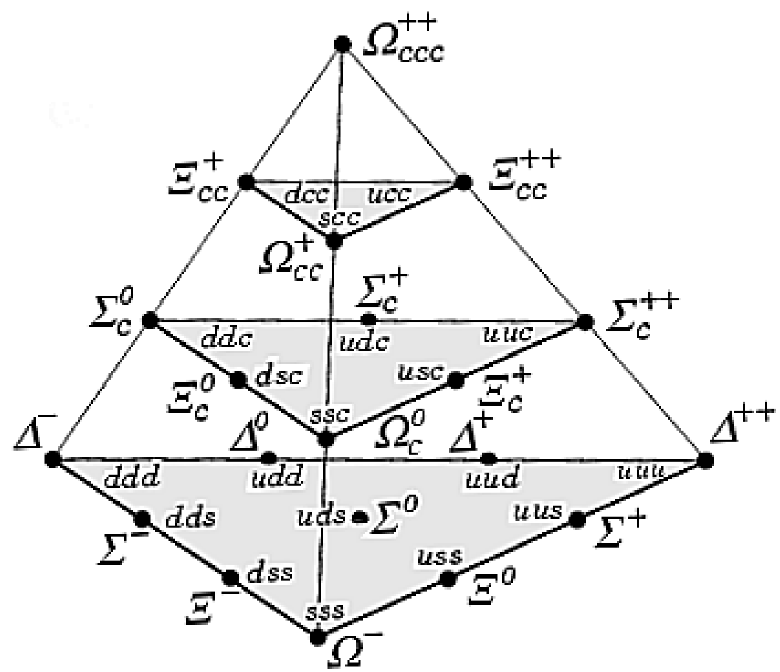
baryons $= qqq, \; qqgq$ or $qqqq\bar{q}$				
symbol	name	el.charge $[Q_e]$	mass $[m_p]$	spin
p	протон	$+1$	1	$1/2$
n	нейтрон	0	1.001	$1/2$
Δ	Δ-изобара	$+2, \pm 1, 0$	1.31	$3/2$
Λ	Λ-гиперон	0	1.19	$1/2$
Σ	Σ-гиперон	$0, \pm 1$	1.27	$1/2$
Ξ	Ξ-гиперон	$0, -1$	1.40	$1/2$
Ω	Ω-гиперон	-1	1.78	$3/2$

Hadrons of Standard Model

Baryonic 20-plets, build up from $\boldsymbol{u}$-, $\boldsymbol{d}$-, $\boldsymbol{s}$- and $\boldsymbol{c}$-quarks.



octet baryons



decuplet baryons

Renormalization:

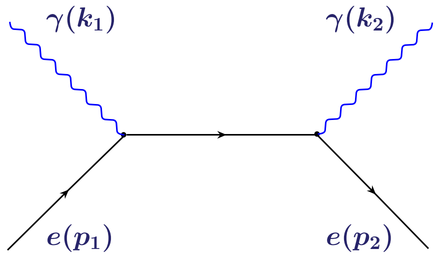
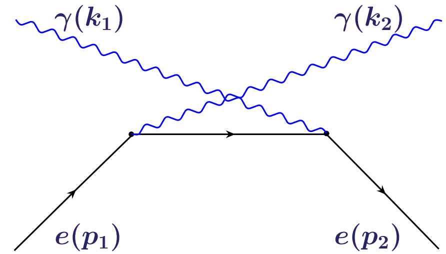
General Scheme

and

Illustration

Radiative Corrections in QFT

Consider Compton scattering in QED:

$M \sim$  $+$  $+ \sim \alpha = \dfrac{e^2}{4\pi}$

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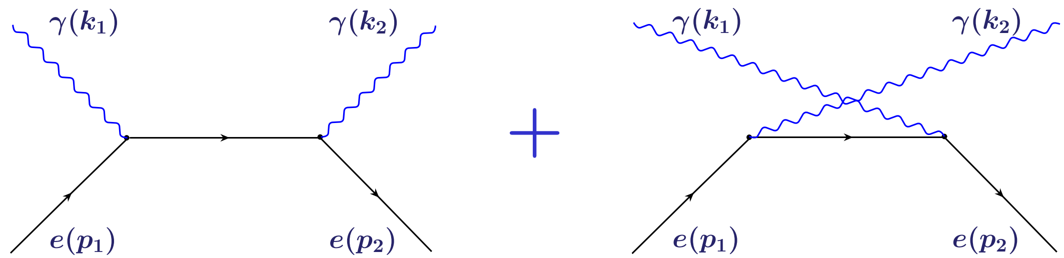

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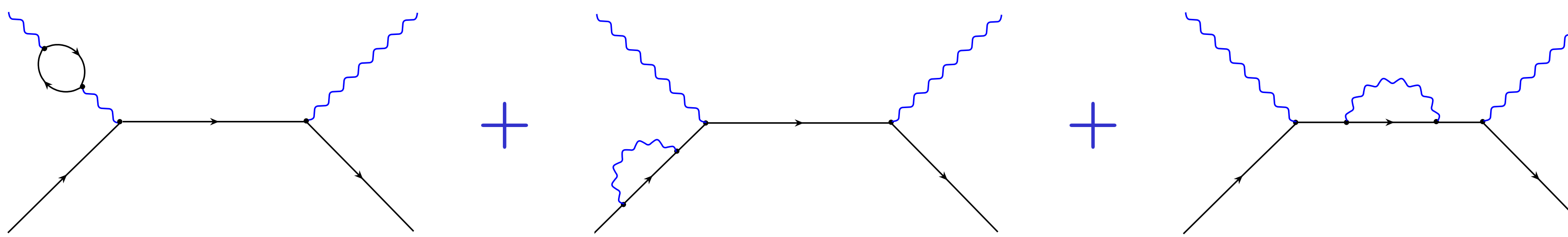

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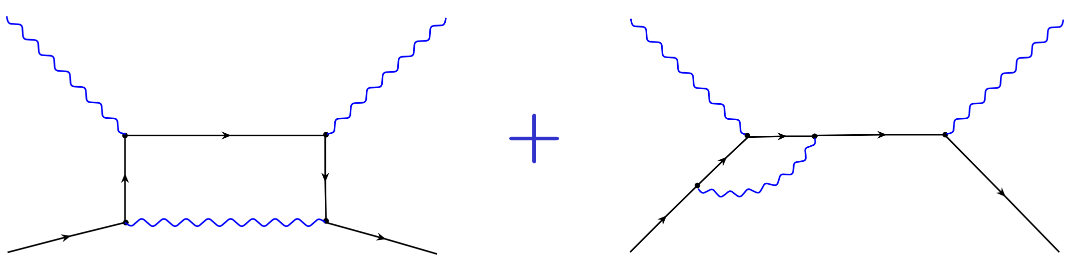
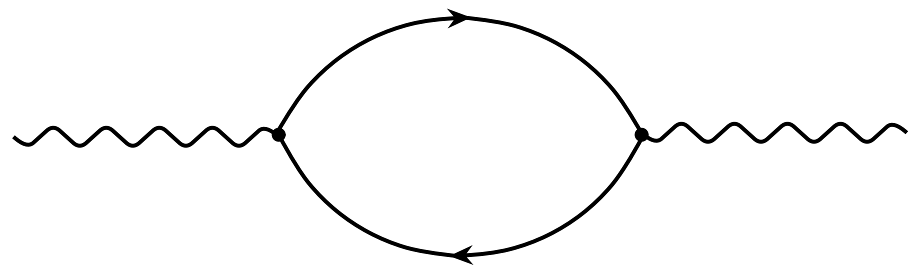


$+ \sim \alpha^2$

$$M = \alpha\, M_1 + \alpha^2\, M_2 + \ldots$$

Radiative Corrections in QFT

$$M = \alpha\, M_1 + \alpha^2\, M_2 + \ldots$$

Contribution from  :

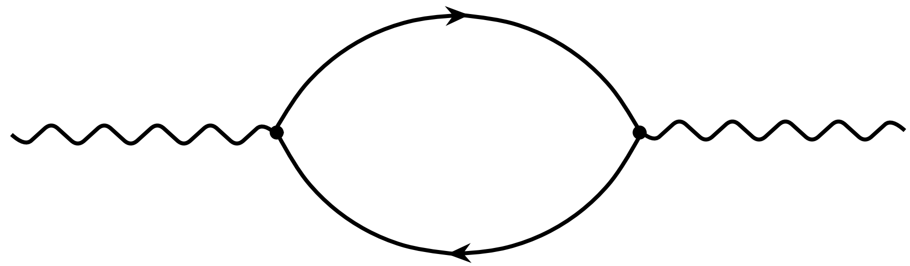
$$\Pi_{\mu\nu}(p) \sim \int \frac{\mathbf{Sp}\left[\gamma_\mu\,(m+\hat{q})\,\gamma_\nu\,(m+\hat{q}-\hat{p})\right]\, d^4q}{[m^2-q^2]\,[m^2-(q-p)^2]}\,.$$

Its UV asymptotics for $q \gg m, p$:

$$\int \frac{q^2\, d^4q}{\left(q^2\right)^2} = \int^{\Lambda} \frac{d^4q}{q^2} \sim \int^{\Lambda} dq^2 \sim \Lambda^2 \to \infty\,.$$

Radiative Corrections in QFT

$$M = \alpha\, M_1 + \alpha^2\, M_2 + \ldots$$

Contribution from  :

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More precisely:

$$\Pi_{\mu\nu}(p) \sim (g_{\mu\nu}p^2 - p_\mu p_\nu)\,\left[\textbf{ln}\frac{\Lambda^2}{p^2} + \textbf{finite part}\right].$$

Radiative corrections in QFT

Compton scattering in QED:

$$M = \alpha\, M_1 + \alpha^2\, M_2 + \ldots$$

More precisely:

$$\Pi_{\mu\nu}(p) \sim \left(g_{\mu\nu}p^2 - p_\mu p_\nu\right)\left[\textbf{ln}\frac{\Lambda^2}{p^2} + \textbf{finite part}\right]$$

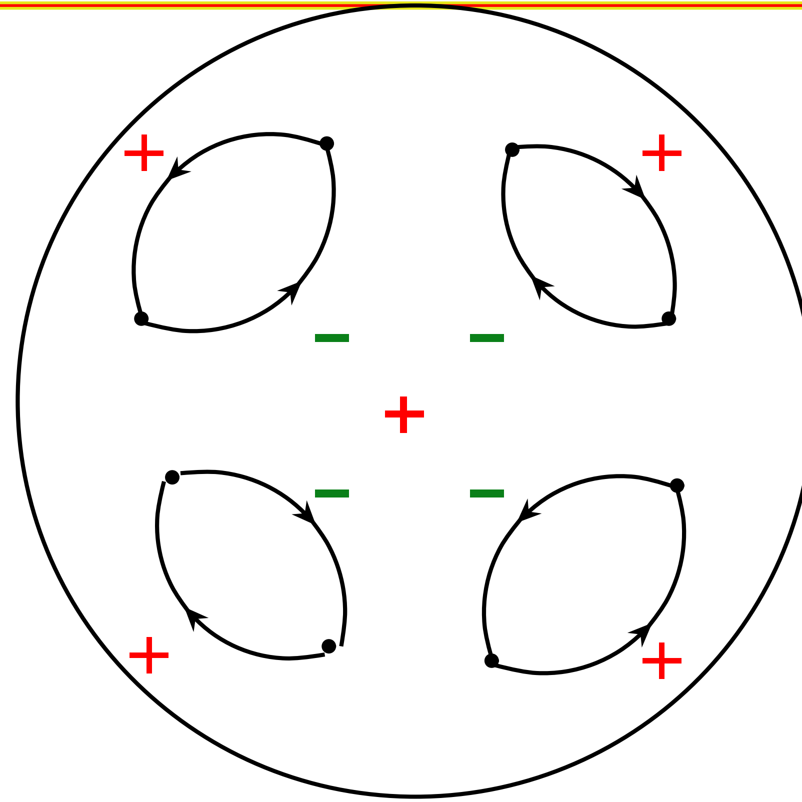
Then Compton scattering amplitude

$$M = \alpha\, R_1 + \alpha^2\left[a\, R_1\, \textbf{ln}\frac{\Lambda^2}{p^2} + R_2\right] + \ldots,$$

where R_i are regular functions of p_i^2, k_i^2 and a is some constant.

Vacuum Polarization

Vacuum in QFT is filled with virtual fields. Insertion **“bare” charge** into vacuum generates the cloud of virtual fields, screening its value to **observed charge**.



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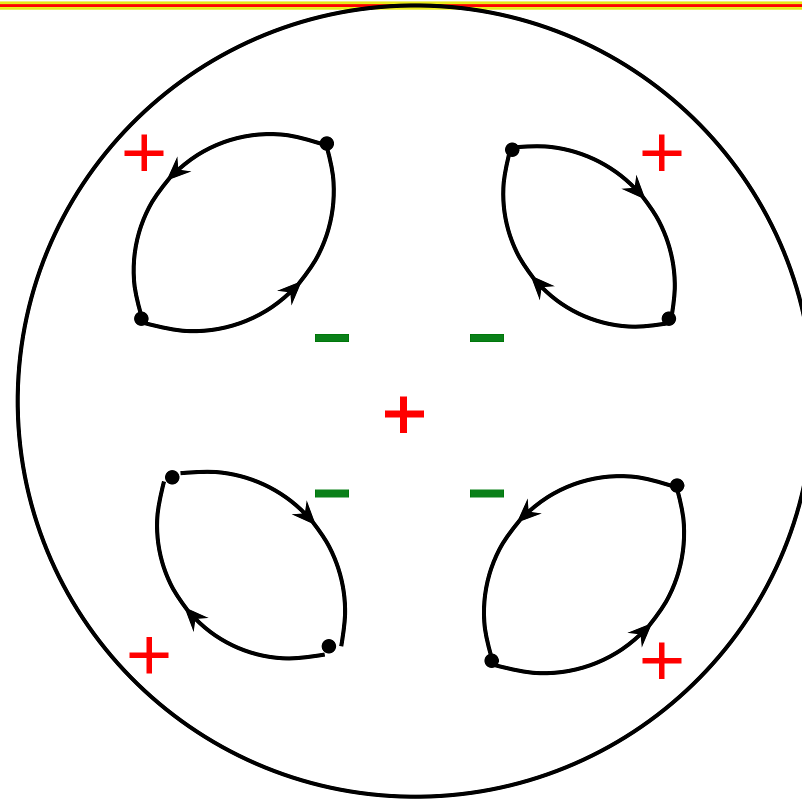
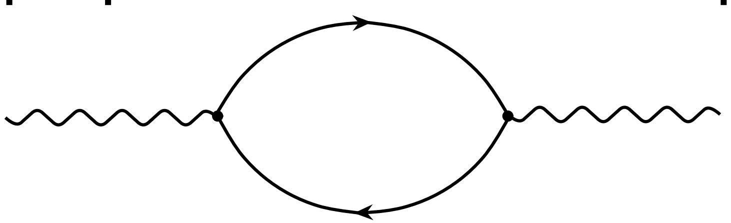


Observed charge

Difference between **"bare"** and **effective** charges is proportional to vacuum polarization:

 $\sim \alpha \ln \frac{\Lambda^2}{p^2}$.

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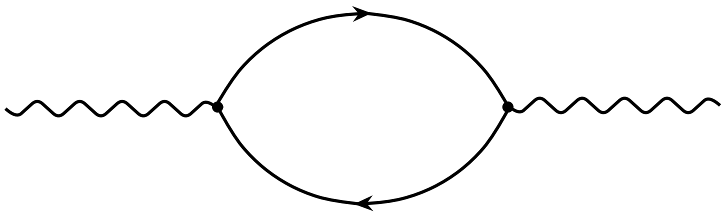 $\sim \alpha\,\ln\frac{\Lambda^2}{p^2}$. Renormalization:

"bare" charge α_{bare} is singular, whereas **effective** charge α is finite **(note auxilary parameter μ^2)**:

$$\alpha_b = \alpha + x\,\alpha^2\,\ln\frac{\Lambda^2}{\mu^2} + \ldots$$

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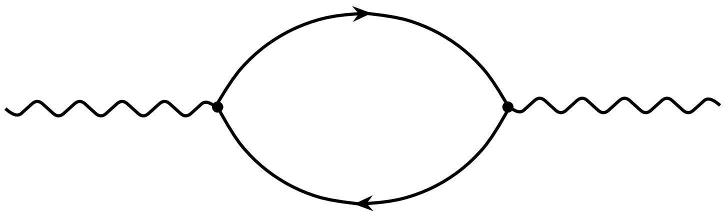

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Rewrite Compton amplitude:

$$M(\ldots\alpha_{\text{b}}\ldots) = \tilde{M}(\ldots\alpha\ldots)$$

$$= \alpha\,R_1 + \alpha^2 \left[a\,R_1 \ln\frac{\Lambda^2}{p^2} + R_2 + x\,R_1 \ln\frac{\Lambda^2}{\mu^2}\right] + \ldots$$

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If we put $x = -a$ then $\tilde{M}$ is finite at $\Lambda \to \infty$:

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We have finite amplitude, but it depends on axillary parameter μ^2. This is fictitious dependence: $\alpha = \alpha(\mu^2)$, so that the physical amplitude $\tilde{M}$ does not depend on μ^2 at all!

Vacuum Polarization

Important: in our simple example all UV divergences (at $\Lambda \to \infty$) are absorbed into redefinition (**renormalization**) of coupling $\alpha_{\mathbf{b}} \to \alpha(\mu^2)$.

Vacuum Polarization

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Bogoliubov&Parasiuk (1957) proved: it is possible to define **renormalization** of fields, couplings and masses

$$\psi_{\mathbf{b}} \;=\; Z_{\psi}^{-1/2}(\mu^2)\,\psi(\mu^2)\,, \qquad \alpha_{\mathbf{b}} = Z_{\alpha}^{-1}(\mu^2)\,\alpha(\mu^2)\,,$$

$$m_{\mathbf{b}} = Z_{m}^{-1}(\mu^2)\,m(\mu^2)$$

in such a way that S-matrix is finite and unique.

Vacuum Polarization

Important: in our simple example all UV divergences (at $\Lambda \to \infty$) are absorbed into redefinition (**renormalization**) of coupling $\alpha_{\mathsf{b}} \to \alpha(\mu^2)$.

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in such a way that S-matrix is finite and unique.

RG is just here: independence of **"bare" quantities** on μ^2 gives us equations for determination dependencies $\psi(\mu^2)$, $\alpha(\mu^2)$, $m(\mu^2)$, $\ldots$ on μ^2.

Renormgroup in QFT:

QED and QCD β-functions
in
1-loop approximation

1-loop RG equation in QCD and QED

1-loop β-function: $\beta_{\text{1-loop}}(\alpha) = b_0\,\alpha^2$

$$\frac{d\alpha(\mu^2)}{d\ln\mu^2} = b_0\,\alpha^2(\mu^2) \qquad \Rightarrow \qquad \alpha(\mu^2) = ???$$

1-loop RG equation in QCD and QED

1-loop β-function: $\beta_{\textbf{1-loop}}(\alpha) = b_0\,\alpha^2$

$$\frac{d\alpha(\mu^2)}{d\ln\mu^2} = b_0\,\alpha^2(\mu^2) \;\Rightarrow\; -\frac{1}{\alpha(\mu^2)} + b_0\ln\mu^2 = const = b_0\ln\Lambda^2$$

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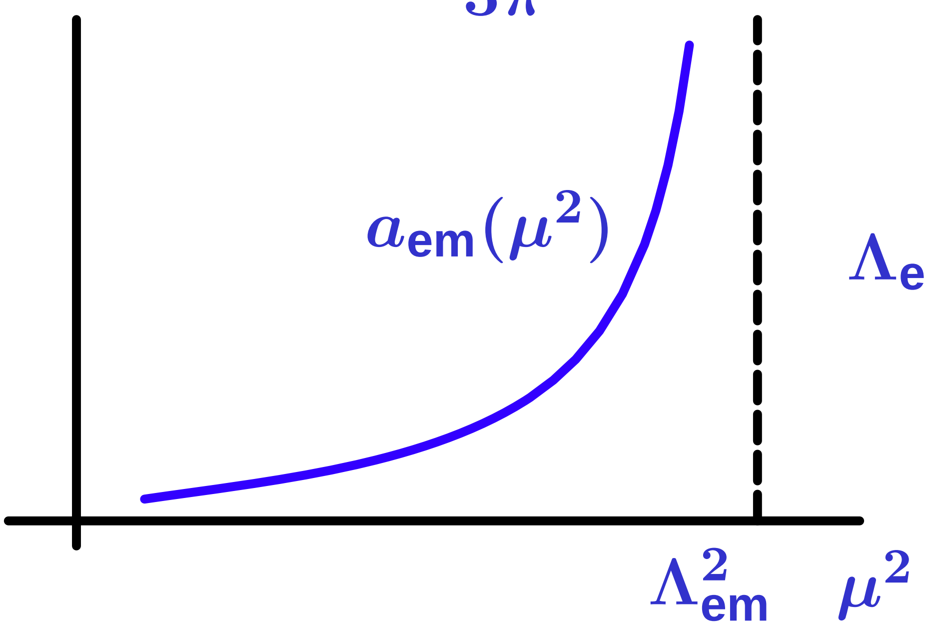
QED case:

$$b_0^{\text{QED}} = \frac{1}{3\pi}\,, \qquad a_{\text{em}}^{\text{1-loop}}(\mu^2) = \frac{3\pi}{\ln\dfrac{\Lambda_{\text{em}}^2}{\mu^2}}$$



$$\Lambda_{\text{em}} \sim m_e\, e^{1/(2\,b_0\,\alpha_0)} \approx 10^{280}\ \text{MeV}$$

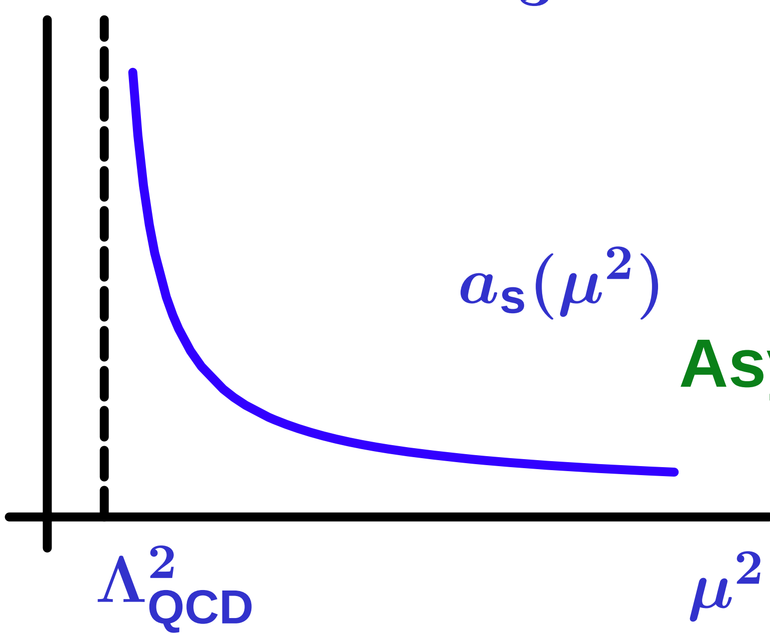
1-loop RG equation in QCD and QED

1-loop β-function: $\beta_{\text{1-loop}}(\alpha) = b_0\,\alpha^2$

$$\frac{d\alpha(\mu^2)}{d\ln\mu^2} = b_0\,\alpha^2(\mu^2) \qquad \Rightarrow \qquad \alpha(\mu^2) = \frac{1}{b_0 \ln\dfrac{\Lambda^2}{\mu^2}}$$

QCD case ($N_c = 3, n_f = 3$)

$$4\pi\, b_0^{\text{QCD}} = -11 + \frac{2}{3}n_f = -9\,, \qquad a_s^{\text{1-loop}}(\mu^2) = \frac{4\pi}{9\ln\dfrac{\mu^2}{\Lambda^2_{\text{QCD}}}}$$



$\Lambda_{\text{QCD}} \approx 300$ MeV

Asymptotic freedom at $\mu^2 \gg 1$

1-loop RG equation in QCD and QED

1-loop β-function: $\beta_{\text{1-loop}}(\alpha) = b_0\,\alpha^2$

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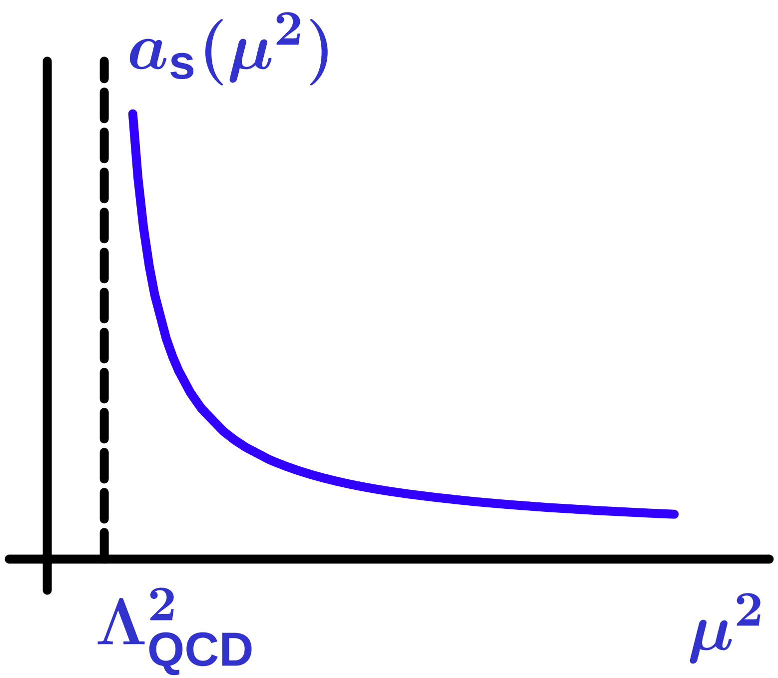

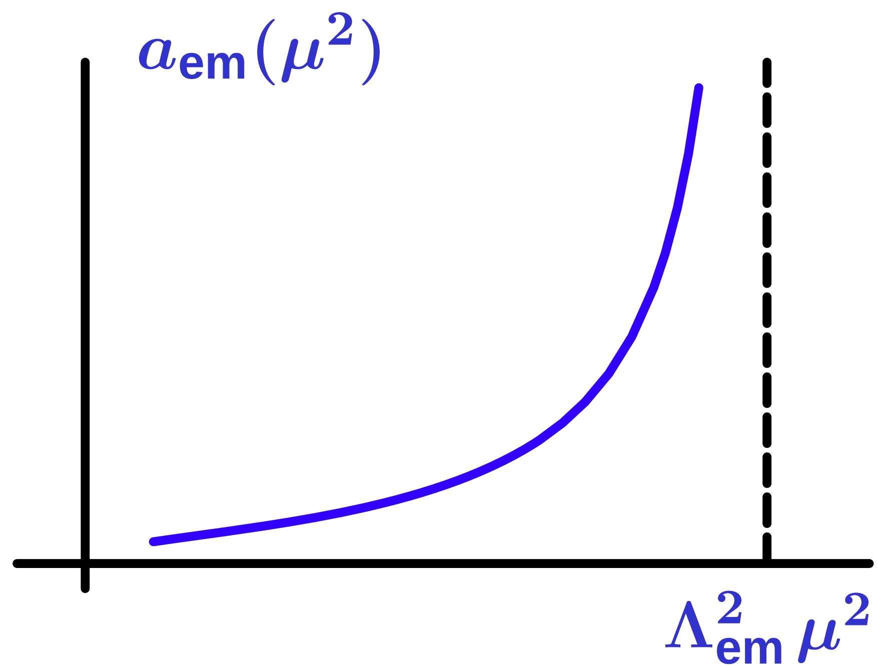

QED PT:

Effective coupling

and

Moscow Zero-Charge

Moscow Zero-Charge problem in QED

Solution

$$\ln x = \int_{\alpha_\Lambda}^{\bar{\alpha}(x;\alpha_\Lambda)} \frac{d\,\alpha}{b_0\,\alpha^2}\,; \quad \bar{\alpha}(x;\alpha_\Lambda) = \frac{\alpha_\Lambda}{1 - b_0\,\alpha_\Lambda \ln x}\,.$$

was obtained by **Landau–Abrikosov–Khalatnikov (1954)** using summation of main logarithms and with nonlocal regularization (parameter Λ).

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Bogoliubov&Shirkov (1955): Perturbative theory is not applicable if $b_0\,\alpha_0 \ln x \gtrsim 1$, that is

$$\sqrt{p^2} \gtrsim m_e\, e^{1/(2\,b_0\,\alpha_0)} = 0.5\, e^{137\cdot 3\,\pi/2}\ \text{MeV} \approx 10^{280}\ \text{MeV}\,.$$

Resolution of Zero-Charge problem in QED

Perturbative theory is not applicable if $\sqrt{p^2} \gtrsim 10^{280}$ **MeV**.
Bogoliubov&Shirkov used 3-loop result

$$\beta(\alpha) = b_0\,\alpha^2\,\left[1 + \frac{3\,\alpha}{4\,\pi} + \frac{3\,\alpha^2}{8\,\pi^2}\left(\frac{8}{3}\zeta(3) - \frac{101}{36}\right)\right]$$

of **Baker&Johnson (1969)** to produce

$$\frac{1}{\beta(\alpha)} = \frac{1}{b_0\,\alpha^2}\,\left[1 - \frac{3\,\alpha}{4\,\pi} + \frac{\alpha^2}{\pi^2}\left(\frac{155}{96} - \zeta(3)\right)\right]$$

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Then at $x \to \infty$ asymptotics is $\bar{\alpha}(x;\alpha) \to \textbf{const} \cdot \ln x$.

No “zero-charge” problem! Instead, strong coupling regime.

Inclusive processes

Bjorken scaling in DIS

and

DGLAP evolution

Deep Inelastic ep-Scattering (DIS)

Bjorken (1968) noted that:

- Structure functions are dimensionless.

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- Then for $Q^2 \gg m_N^2$ they can depend only on dimensionless parameter $x_{\text{Bj}} = Q^2/(2\,P\cdot q)$:

$$F_i(x, Q^2)_{Q^2 \to \infty} \longrightarrow F_i(x)$$

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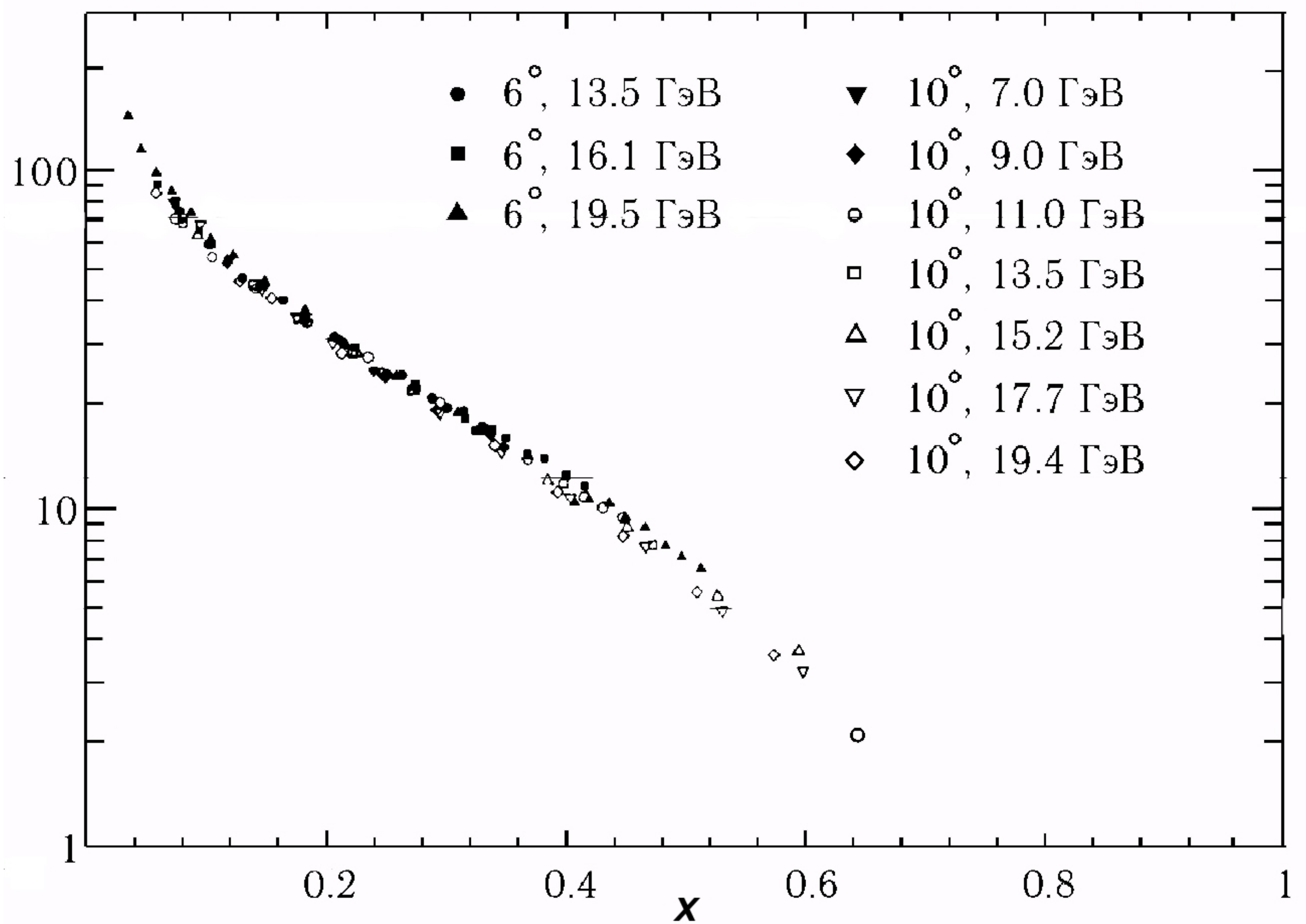
This is just **Bjorken scaling**, which dictates DIS cross-section to be:

$$Q^4 \left[1 + \left(1 - \frac{Q^2}{x\,s}\right)^2\right]^{-1} \frac{d^2\sigma}{d\,x\,d\,Q^2} = 2\,\pi\,\alpha^2 \sum_i e_i^2\, f_i(x)$$

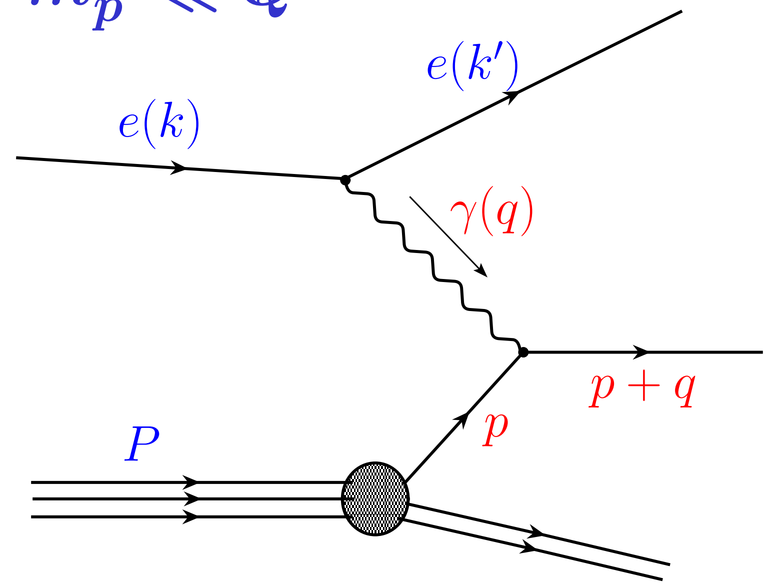
Deep Inelastic ep*-Scattering (DIS)*

Bjorken (1968) by hands reoriented detector in SLAC to measure ep-scattering in **non-forward direction**.



Parton model description of ep-DIS

Feynman (1969) proposed a parton model explanation for scaling in ep-DIS. Kinematics: $P^2 = m_p^2 \ll Q^2$



$$\hat{t} = q^2 = -Q^2;$$
$$\hat{s} = (p+k)^2 = 2\,\xi\,P\cdot k = \xi\,s;$$
$$0 = (p+q)^2 = 2\,\xi\,P\cdot q - Q^2;$$
$$\xi = x \equiv \frac{Q^2}{2\,P\cdot q}.$$

Cross-section on parton level (Mott formula):

$$\frac{d\sigma}{d\hat{t}} = \frac{2\,\pi\,\alpha^2\,e_i^2}{\hat{t}^2}\left(1 + \frac{(\hat{s}+\hat{t})^2}{\hat{s}^2}\right)$$

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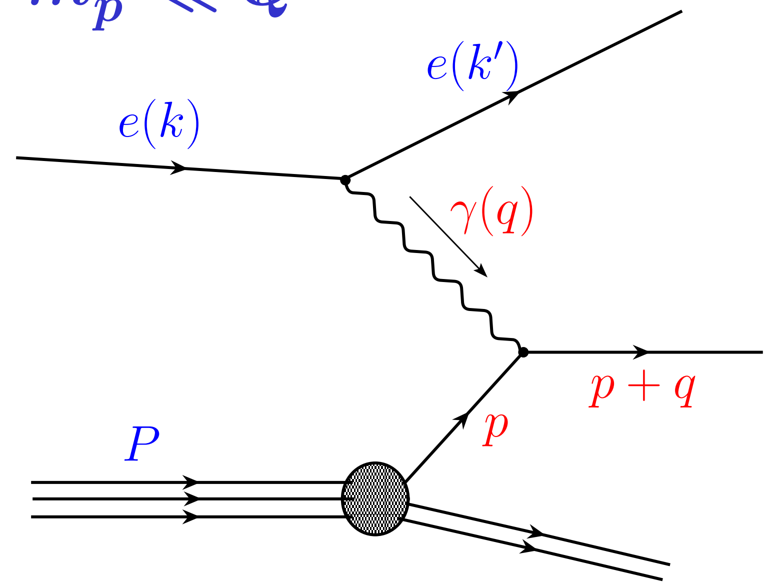

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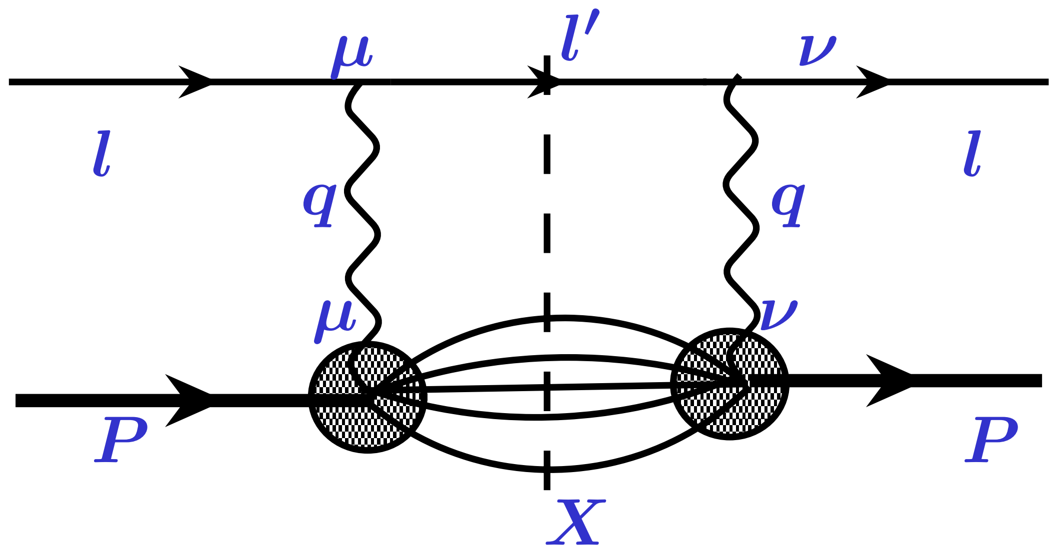
Each parton enters with probability $f_i(\xi = x)$

$$\frac{d^2\sigma}{d\,x\,d\,Q^2} = \sum_i f_i(x)\,e_i^2\,\frac{2\,\pi\,\alpha^2}{Q^4}\left[1 + \left(1 - \frac{Q^2}{x\,s}\right)^2\right]$$

Partons inside hadron are free! Why?

Description of ep-DIS in QCD

DIS: General representation of cross section



Cross section can be represented as:

$$\frac{d\sigma}{d\Omega dE'} \sim L^{\mu\nu} H_{\mu\nu}$$

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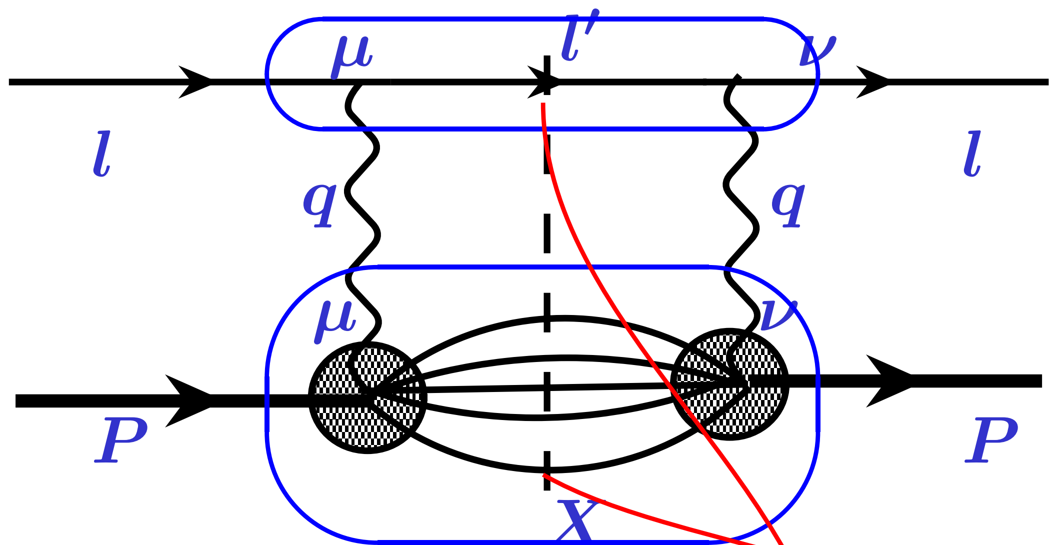

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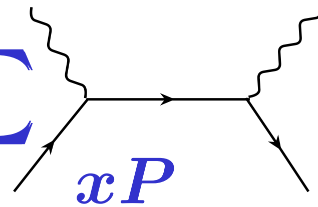
$$L^{\mu\nu} = (\bar{u}(l')\gamma^{\mu}u(l))^{*}(\bar{u}(l')\gamma^{\mu}u(l))$$

Parton distributions in nucleon

Hadron tensor can be represented as:

$$H_{\mu\nu} \sim \sum_i$$



$$\otimes \quad f_i(x, Q^2)$$

where $f_i(x, Q^2)$ – parton distribution function (PDF).

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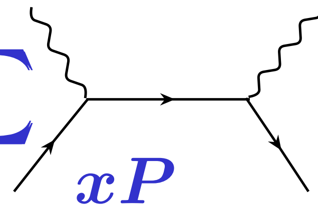

$$\otimes \quad f_i(x, Q^2)$$

Using operator product expansion we can determine PDF moments:

$$M_N(Q^2) = \int_0^1 f(x, Q^2)\, x^N\, dx = \frac{i^N}{2(np)^{N+1}} \langle P | \bar{q}\, \hat{n}\, (nD)^N\, q | P \rangle\,,$$

where $n^2 = 0$.

Parton distributions in nucleon

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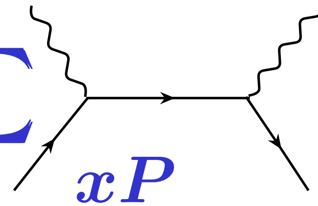

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where $n^2 = 0$. One more time: **All nonperturbative info about nucleon is hidden in the matrix element** $\langle P | \bar{q}\, \hat{n}\, (nD)^N\, q | P \rangle$

DGLAP evolution equation

Composite operators, related to moments $M_N(Q^2)$, are multiplicatively renormalizable. Then we have RG equation:

$$\frac{d\, M_N(\mu^2)}{d \ln \mu^2} = \frac{\alpha_s}{4\,\pi}\,\gamma_0(N)\, M_N$$

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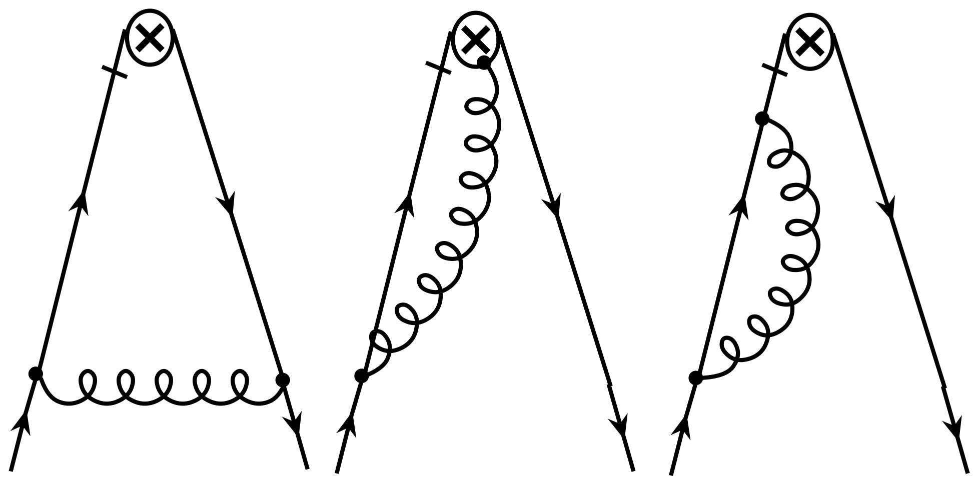
This generates DGLAP equation for PDF:

$$\mu^2\,\frac{d\,f(x;\mu^2)}{d\ln\mu^2} = \frac{\alpha_s}{4\,\pi}\int_0^1\int_0^1 \delta(x-yz)\,P(z)\,f(y;\mu^2)\,dz\,dy\,;$$

$$\gamma_N^0 = \int_0^1 P(z)\,z^N\,dz\,.$$

Leading order in α_s

In $O(\alpha_s)$-order we have 3 diagrams:



Coefficients at $\ln\left(\mu^2/p^2\right)$ give us anomalous dimensions:

$$\gamma_0(N) = C_F\Big[\frac{2}{(N+1)(N+2)} - 4\sum_{j=2}^{N+2}\frac{1}{j} + 1\Big]\,;$$

$$P(x) = C_F\Big[2(1-x)+4\Big(\frac{x}{1-x}\Big)_+ -\delta(1-x)\Big] = 2C_F\Big(\frac{1+x^2}{1-x}\Big)_+$$

Scaling violations in QCD

DGLAP equation solution for moments is simple:

$$M_N(\mu^2) = M_N(\mu_0^2)\left(\frac{\alpha_s(\mu^2)}{\alpha_s(\mu_0^2)}\right)^{\gamma_N^0/b_0}$$

Important: $\gamma_0(0) = 0$!

Why?

Scaling violations in QCD

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Important: $\gamma_0(0) = 0$!
Vector current is conserved!
Important: $\gamma_0(N \geq 1) > 0$!
Consequences:
$M_N(\mu^2 \to \infty) = ?$

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Here we show evolution of $\delta(x - 0.5)$ as $\tau = \frac{C_F}{b_0}\ln\frac{\alpha_0}{\alpha}$ goes from 0 to 3. ☛ ⊙

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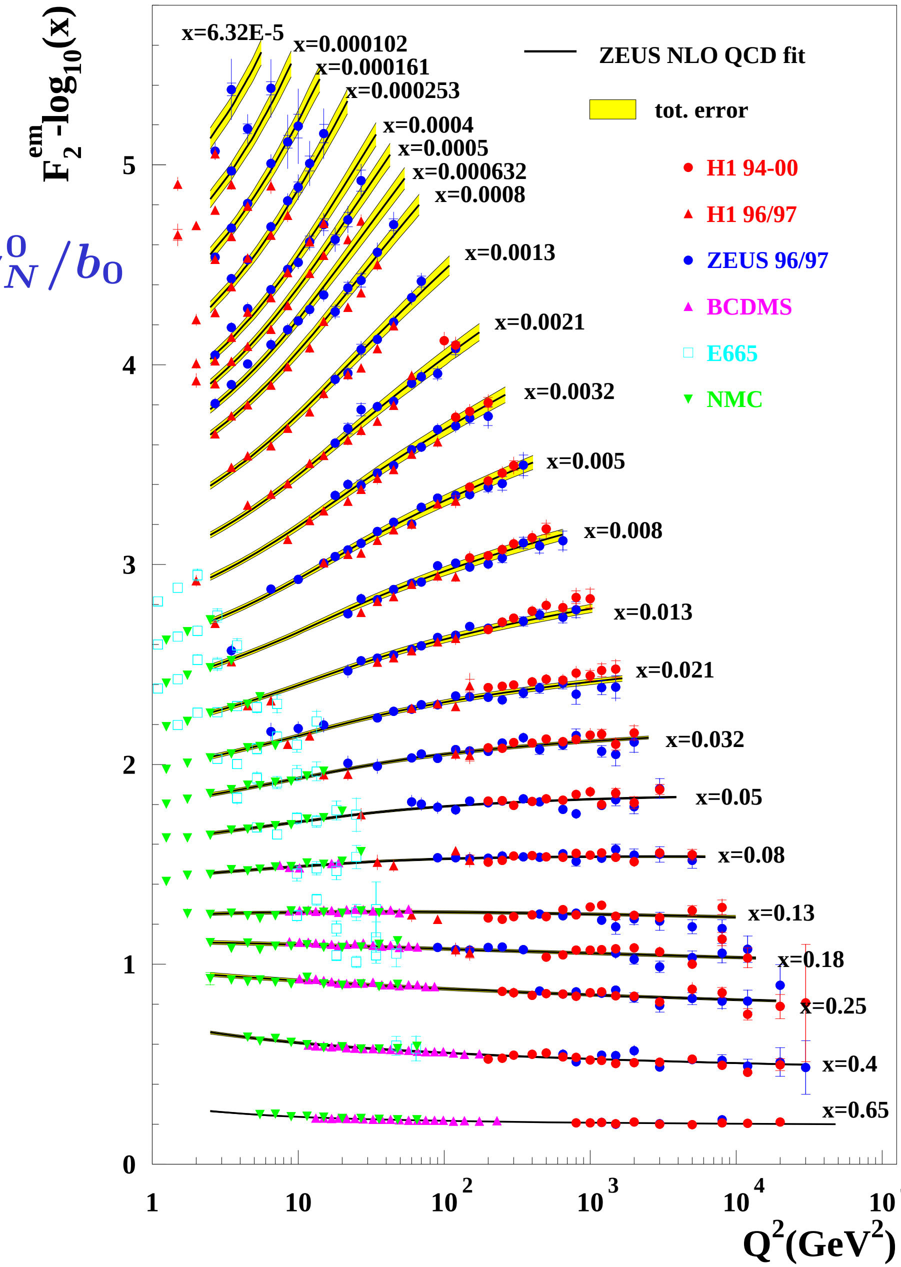
Logarithmic dependence of $\alpha_s(\mu^2)$ on μ^2 generates Logarithmic scaling violation!



Annihilation

$$e^+e^- \rightarrow \textbf{\textit{hadrons}}$$

in

QCD

Vector current correlator $\Pi_{\mu\nu}$

Lorentz invariance and vector current conservation dictate

$$\Pi_{\mu\nu}(q) = i\int d^4x\, e^{iqx}\langle 0\big|T\left[J^{\mu}(x)J_{\nu}(0)\right]\big|0\rangle = \left[q_{\mu}\, q_{\nu} - g_{\mu\nu}\, q^2\right]\Pi(q)$$

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Question: What is dimensionality of $\Pi(q)$?

$$\begin{aligned} D[J^\mu(x)] &= D[\psi]^2 = M^3 \\ D[\Pi_{\mu\nu}(q)] &= D[J^\mu(x)]^2 - M^4 = M^2 \\ D[\Pi(q)] &= ? \end{aligned}$$

Vector current correlator $\Pi_{\mu\nu}$

Lorentz invariance and vector current conservation dictate

Inserting $\hat{1}$ in between currents we obtain

$$\mathbf{\Pi}(\boldsymbol{q}) = \frac{-i}{3q^2}\sum_{X(p)}\int_0^{\infty} dt\, e^{iq_0 t}\int d^3\vec{x}\, e^{-i\vec{q}\vec{x}}\langle 0\big|J^{\mu}(x)\big|X(p)\rangle\langle X(p)\big|J^{\dagger}_{\mu}(0)\big|0\rangle$$

$$+\frac{-i}{3q^2}\sum_{X(p)}\int_{-\infty}^{0} dt\, e^{iq_0 t}\int d^3\vec{x}\, e^{-i\vec{q}\vec{x}}\langle 0\big|J^{\dagger}_{\mu}(0)\big|X(p)\rangle\langle X(p)\big|J^{\mu}(x)\big|0\rangle$$

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$$+ \frac{-i}{3q^2} \sum_{X(p)} \int_{-\infty}^0 dt\, e^{iq_0 t} \int d^3\vec{x}\, e^{-i\vec{q}\vec{x}} \langle 0 \big| J^\dagger_\mu(0) \big| X(p) \rangle \langle X(p) \big| J^\mu(x) \big| 0 \rangle$$

$$= \frac{-i\,(2\pi)^3}{3q^2} \sum_{X(p)} \delta(\vec{p} - \vec{q})\, \theta(p_0) \left| \langle 0 \big| J_\mu(0) \big| X(p) \rangle \right|^2$$

$$\times \int_0^\infty dt \left[e^{i(q_0 - p_0)t} + e^{-i(q_0 + p_0)t} \right]$$

Vector current correlator $\Pi_{\mu\nu}$

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We have the following identities

$$\int_0^\infty dt\, e^{\pm i\alpha t} = \pi\,\delta(\alpha) \pm i\,\mathcal{P}\frac{1}{\alpha}$$

Vector current correlator $\Pi_{\mu\nu}$

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$$\int_0^\infty dt\, e^{\pm i\alpha t} = \pi\,\delta(\alpha) \pm i\,\mathcal{P}\frac{1}{\alpha}$$

After all substitutions:

$$\mathrm{Im}\Pi(q^2) = -\pi \frac{(2\pi)^3}{3q^2} \sum_{X(p)} \delta(\vec{p}-\vec{q})\delta(p_0 - |q_0|) \Big|\langle 0|J_\mu(0)|X(p)\rangle\Big|^2$$

Vector current correlator $\Pi_{\mu\nu}$

So, we have $\frac{1}{\pi}\,\mathbf{Im}\,\Pi(q^2) = \rho(q^2)\theta(|q_0|) = \rho(q^2)$ with

$$\rho(q^2)\theta(q_0) = \frac{-(2\pi)^3}{3q^2}\sum_{X(p)}\delta^{(4)}(q-p)\theta(p_0)\Big|\langle 0|J_\mu(0)|X(p)\rangle\Big|^2$$

Relation with cross section $e^+e^- \to hadrons$

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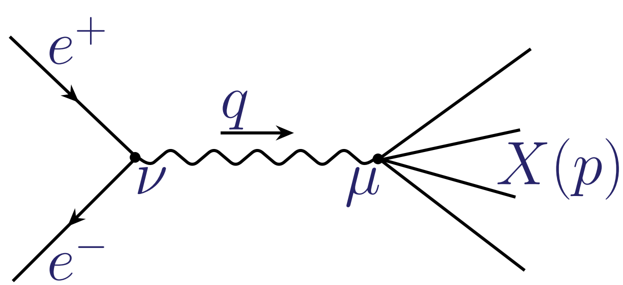
Important! This function naturally appears in 1-photon QED description of $e^+e^- \to$ hadrons:



$$\bar{u}(k)\gamma_\mu u(k')\frac{ie^2}{q^2}\langle X(p)|J_\mu(q)|0\rangle$$

Relation with cross section $e^+e^- \longrightarrow hadrons$

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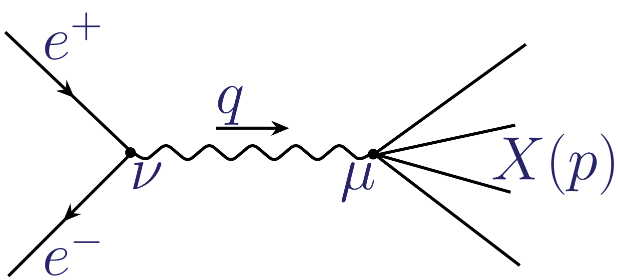
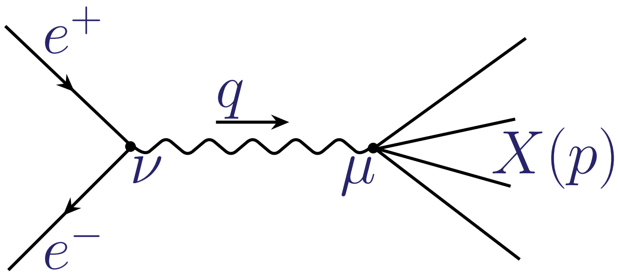


$$\sigma_{\mathrm{had}}(s) = \frac{16\,\pi^3\,\alpha^2}{s}\rho(s)$$

Relation with cross section $e^+e^- \longrightarrow$ *hadrons*

In 1-photon approximation of QED:



$$\sigma_{\text{had}}(s) = \frac{16\,\pi^3\,\alpha^2}{s}\rho(s) = \frac{4\pi\alpha^2}{3\,s}R(s)$$

Here we explicitly extracted as a factor cross-section $\sigma_{\mu^+\mu^-}(s) = 4\pi\alpha^2/(3\,s)$ of the process $e^+e^- \to \mu^+\mu^-$.

Relation with cross section $e^+e^- \longrightarrow hadrons$

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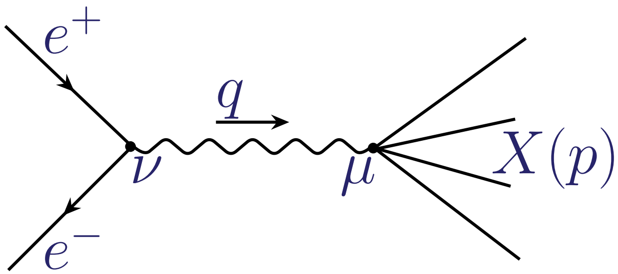

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$$R(s) = \frac{\sigma(e^+e^- \to \textbf{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}\,; \quad \rho(s) = \frac{R(s)}{12\pi^2}$$

Relation with cross section $e^+e^- \longrightarrow hadrons$

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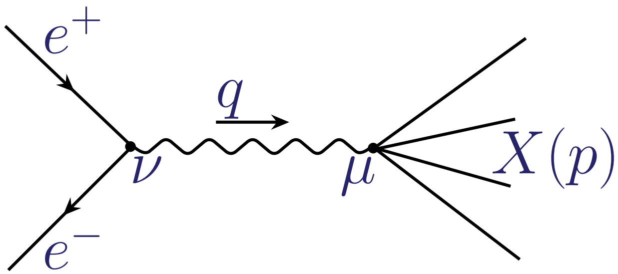

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$$R(s) = \frac{\sigma(e^+e^- \to \textbf{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}\,; \quad \rho(s) = \frac{R(s)}{12\pi^2}$$

Question: What is $R(s)$ in QCD at tree level?

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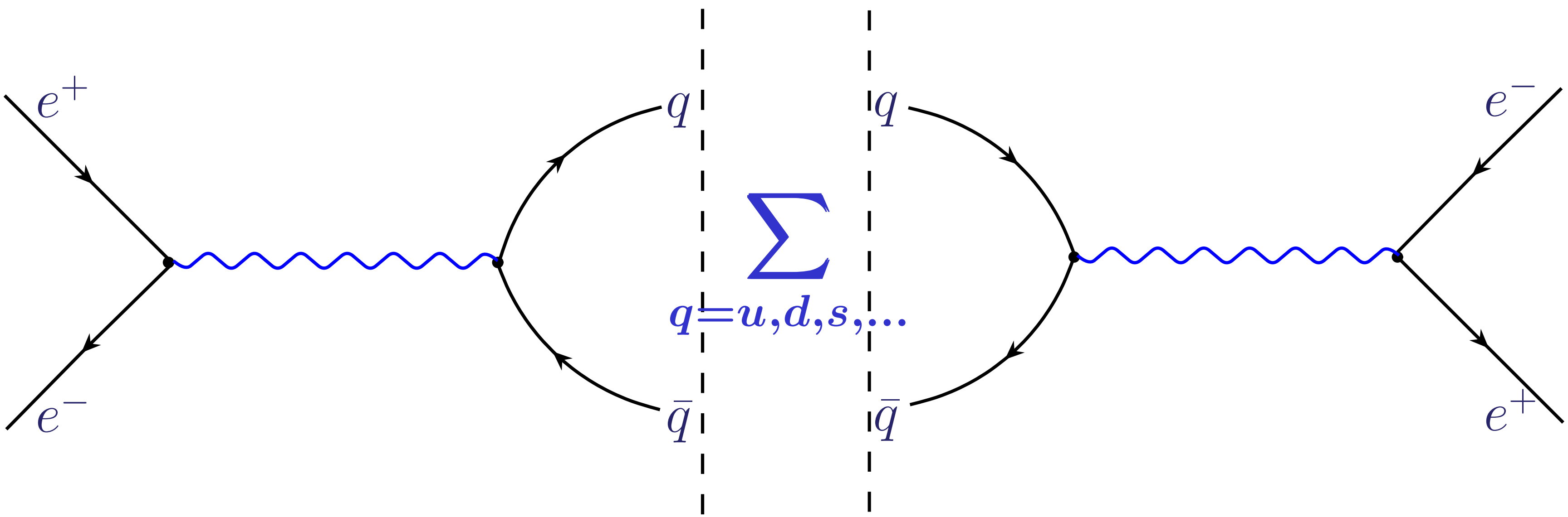

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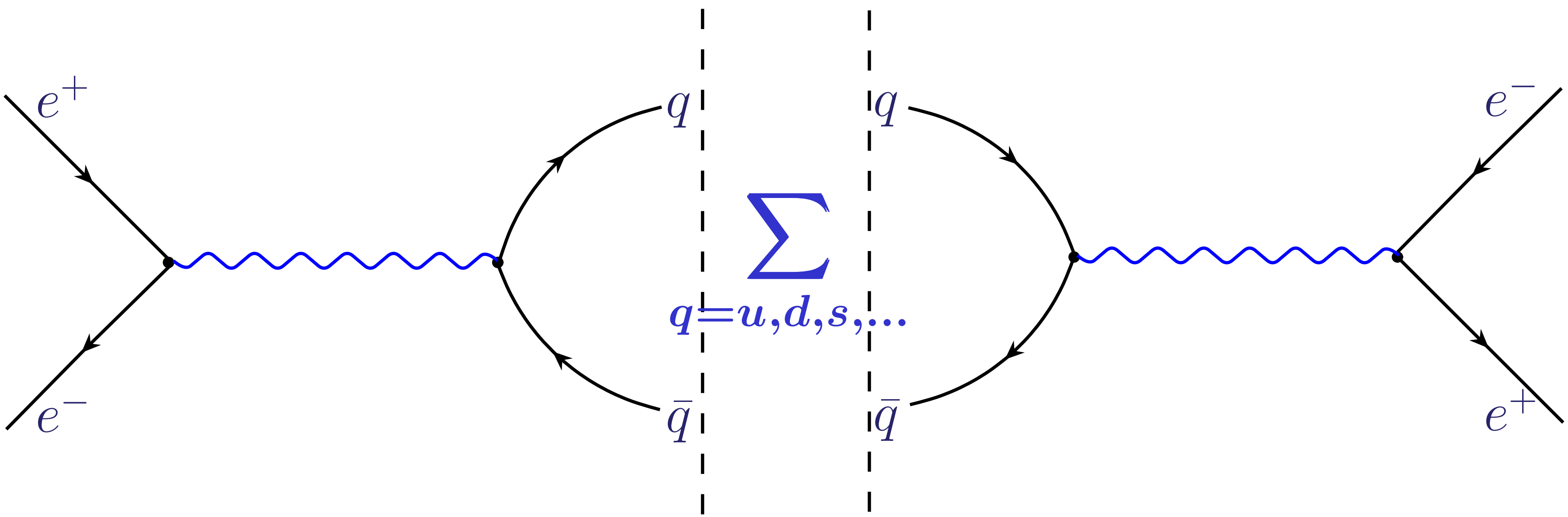

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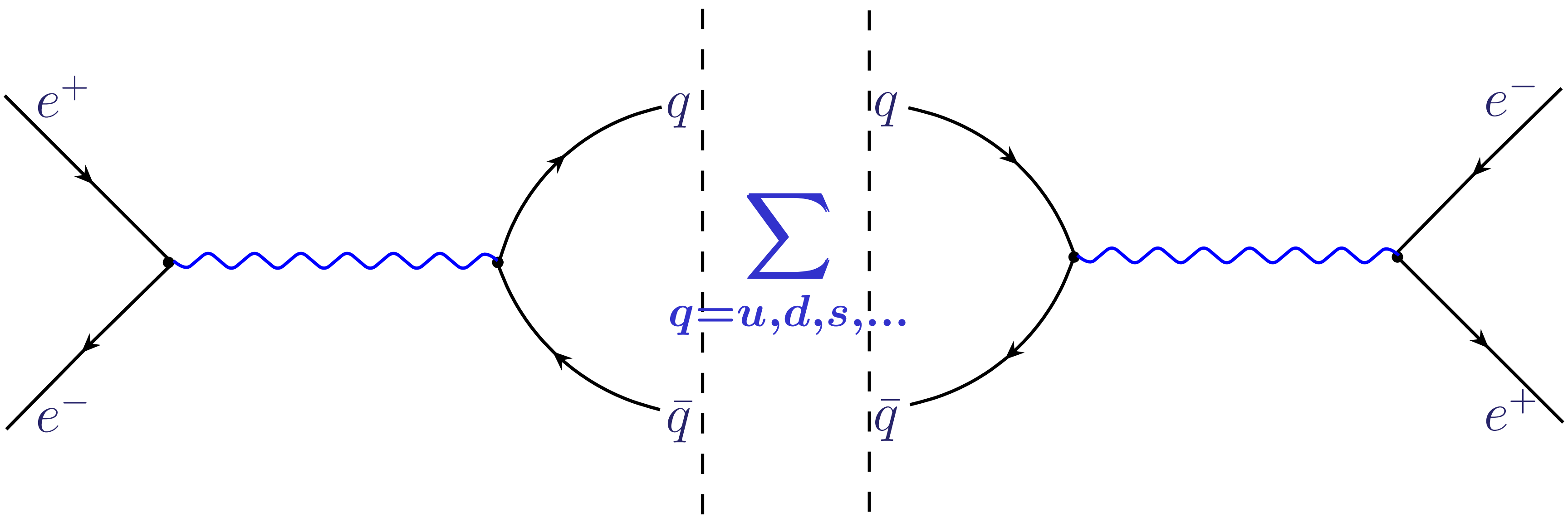

$$R^{\text{tree}}_{\text{QCD}}(s \leq m_c^2) = N_c \sum_{q=u,d,s} e_q^2 = ?$$

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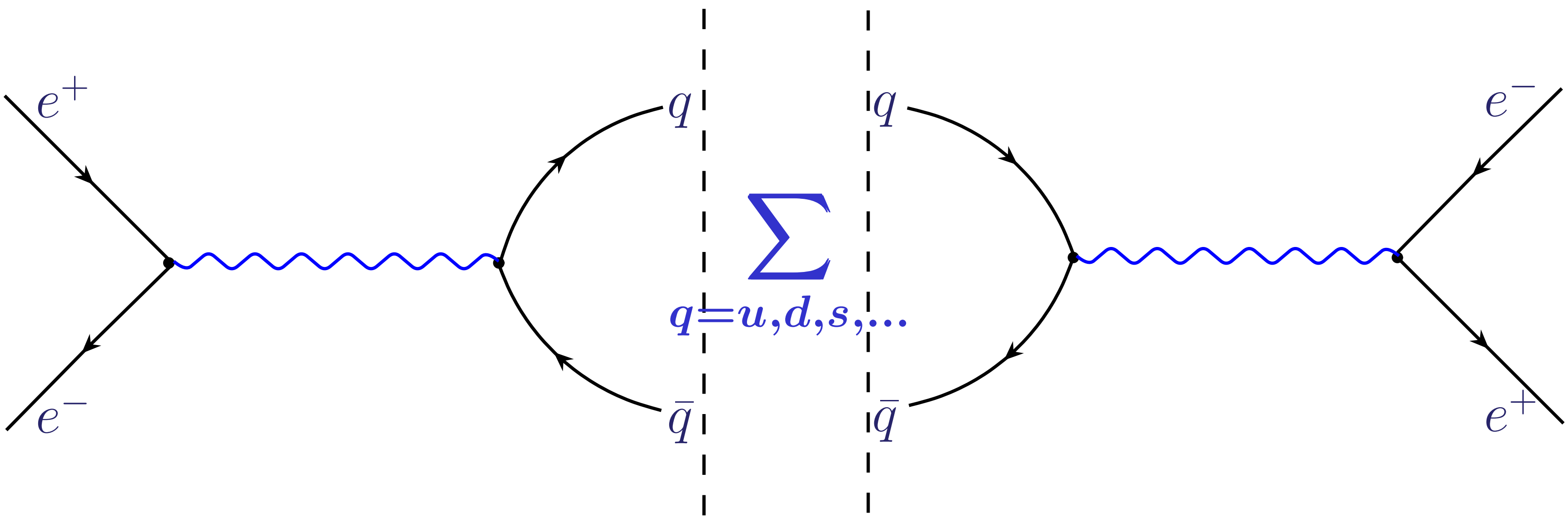

$$R_{\text{QCD}}^{\text{tree}}(s \leq m_c^2) = N_c \sum_{q=u,d,s} e_q^2 = 3\left(\frac{1}{9} + \frac{1}{9} + \frac{4}{9}\right) = 2$$

e^+e^- Annihilation to hadrons

- **Appelquist–Georgi–Zee (1973)** calculated corrections to cross-section:

$$R = \frac{\sigma(e^+e^- \to \textbf{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = N_c \sum_f e_f^2 \left(1 + \alpha_s/\pi\right) \, .$$

For three light quarks
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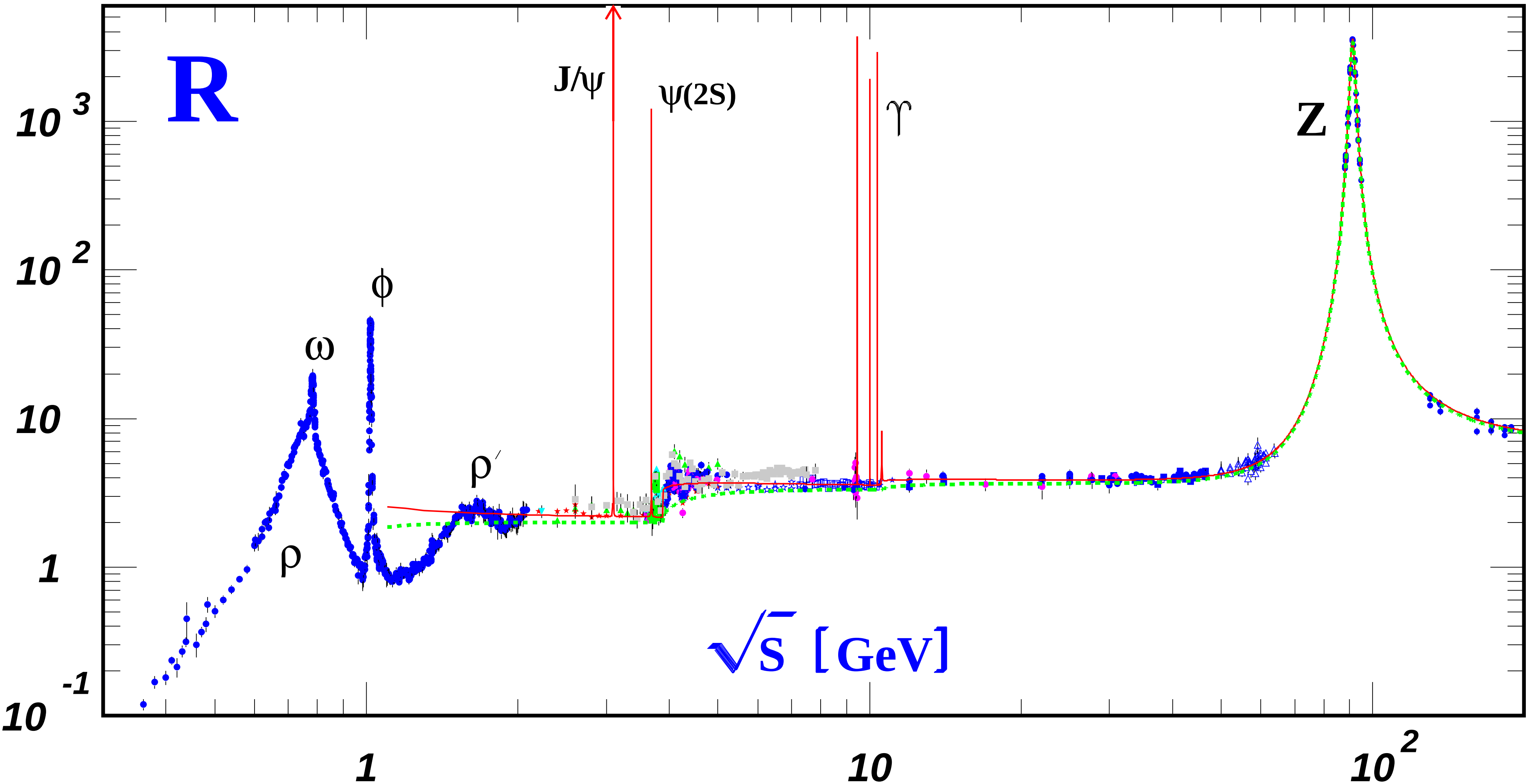
- At **Triest Workshop (spring, 1974)** B. Richter informed that R goes through 2 without any hint to flatness $\Rightarrow$ Pessimism about QCD.

e^+e^- Annihilation to hadrons

- It soon became clear that it was related to new resonances due to c-quarks, namely J/ψ-meson:



Appearance and recognition of QCD

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Appearance and recognition of QCD

- **Asymptotic freedom** explains short-distance **scaling**;
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- **Using QCD one may use** perturbation theory (PT) at short distances and calculate a lot of processes.

Nobel Prize'04 publications

- **Asymptotic freedom**:
  - **D. J. Gross and F. Wilczek**, "Ultraviolet behavior of non-Abelian gauge theories", Phys. Rev. Lett. 30 (1973) 1343;
  - **H. D. Politzer**, "Reliable perturbative results for strong interactions?", Phys. Rev. Lett. 30 (1973) 1346.
- **Asymptotic form** of flavor-nonsinglet Structure Functions: **D. J. Gross and F. Wilczek**, "Asymptotically Free Gauge Theories. I", Phys. Rev. D8 (1973) 3633.
- **Asymptotic form** of flavor-singlet Structure Functions: **D. J. Gross and F. Wilczek**, "Asymptotically Free Gauge Theories. II", Phys. Rev. D9 (1974) 980.

The End
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Intro: PT in QCD

- coupling $\alpha_s(\mu^2) = (4\pi/b_0)\, a_s(L)$ with $L = \ln(\mu^2/\Lambda^2)$

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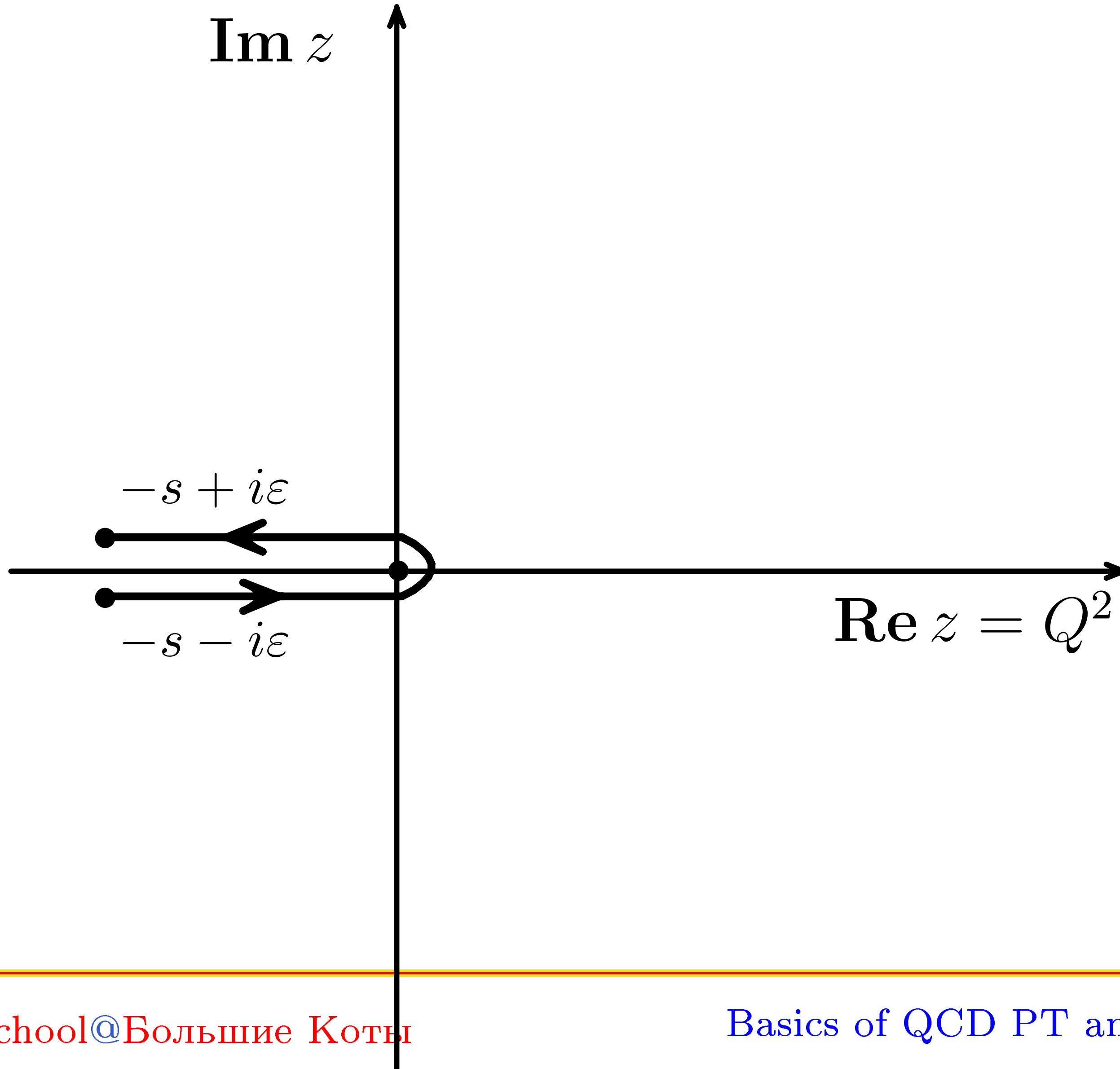
- PT series: $D(L) = 1 + d_1 a_s(L) + d_2 a_s^2(L) + \ldots$

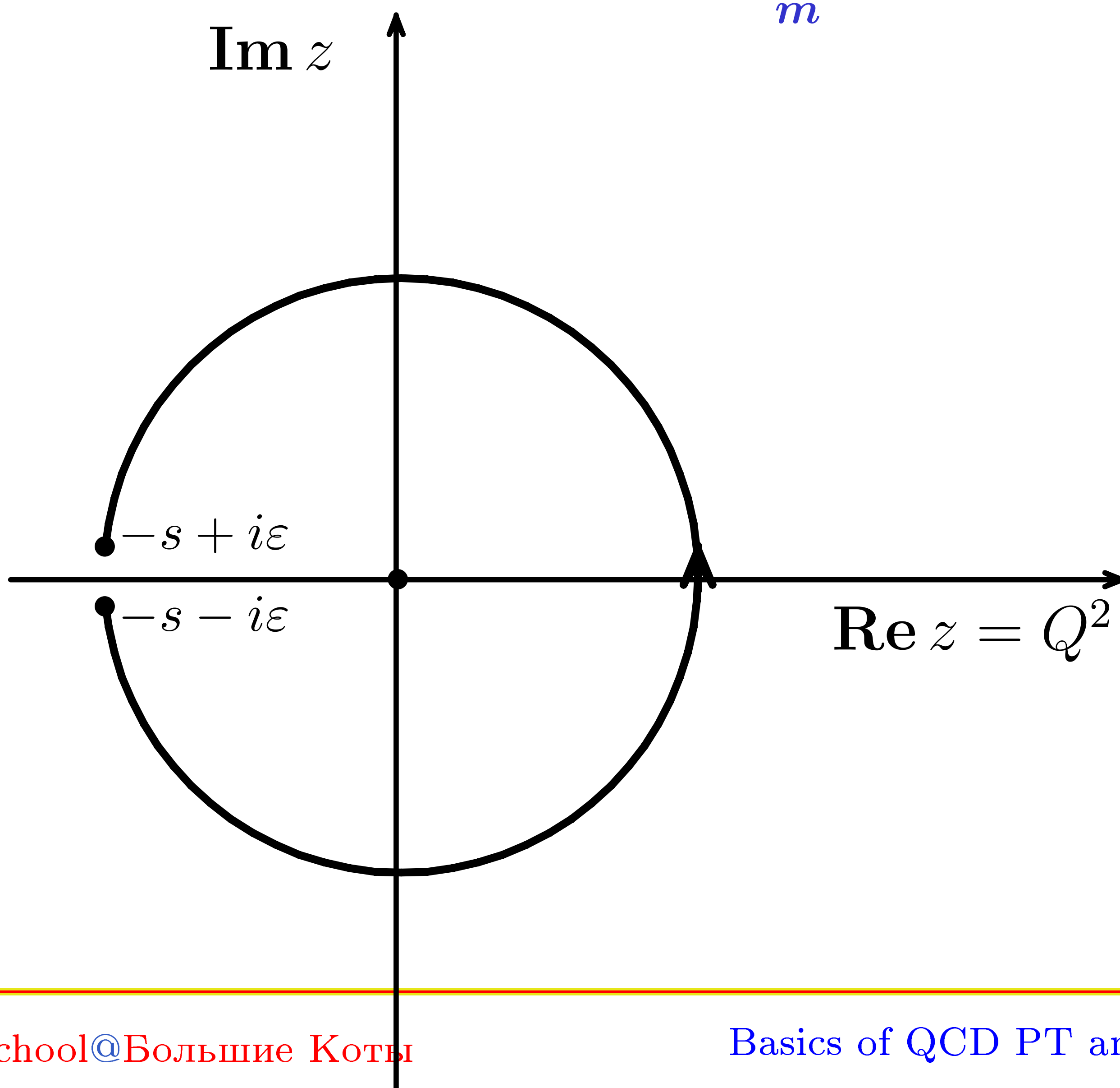
Problem in QCD PT: Minkowski region?

Quantities in Minkowski region $= \oint f(z)D(z)dz$.



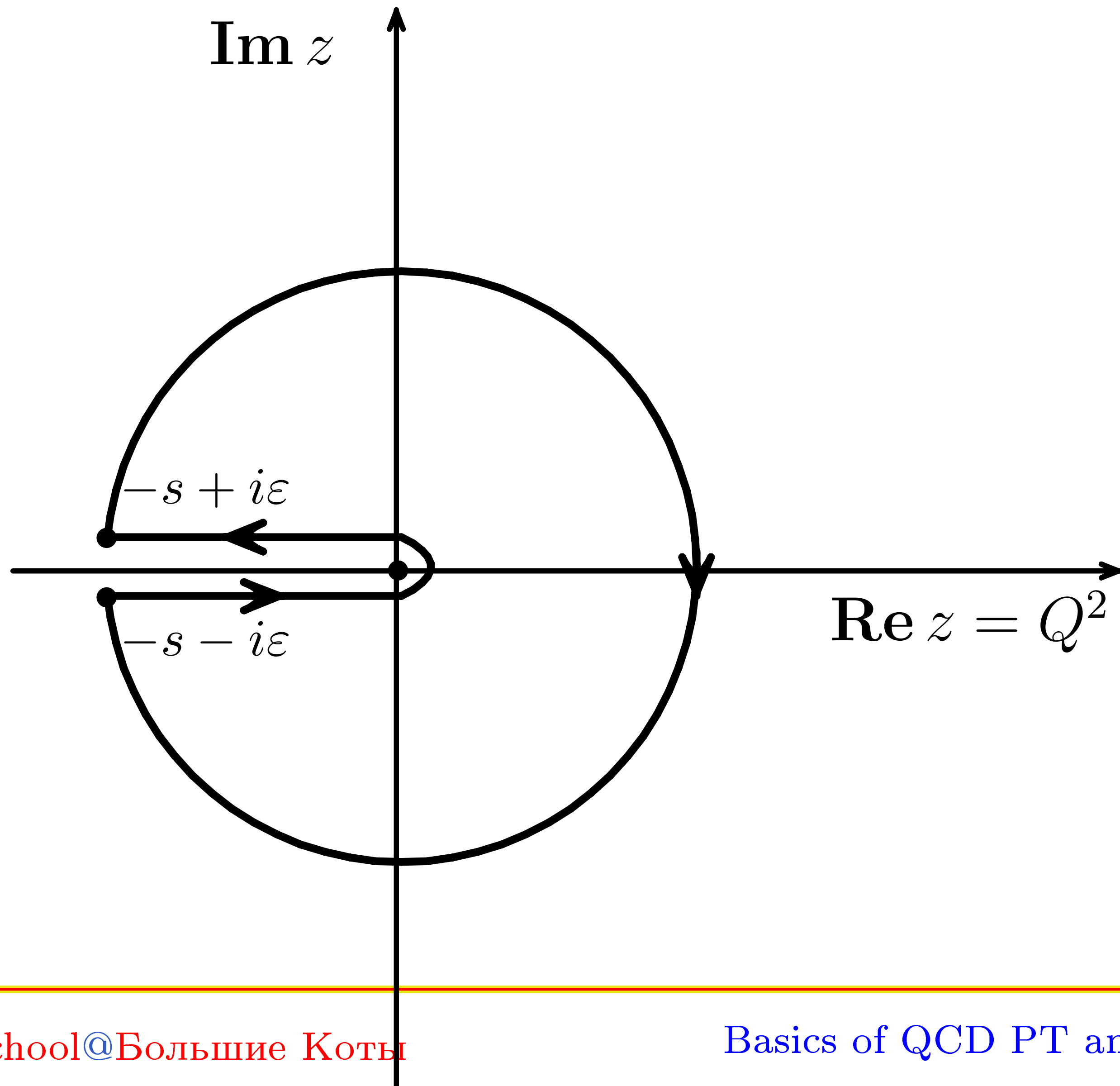
Problem in QCD PT: Minkowski region?

In $\oint f(z)D(z)dz$ one uses $D(z) = \sum_m d_m \alpha_s^m(z)$.



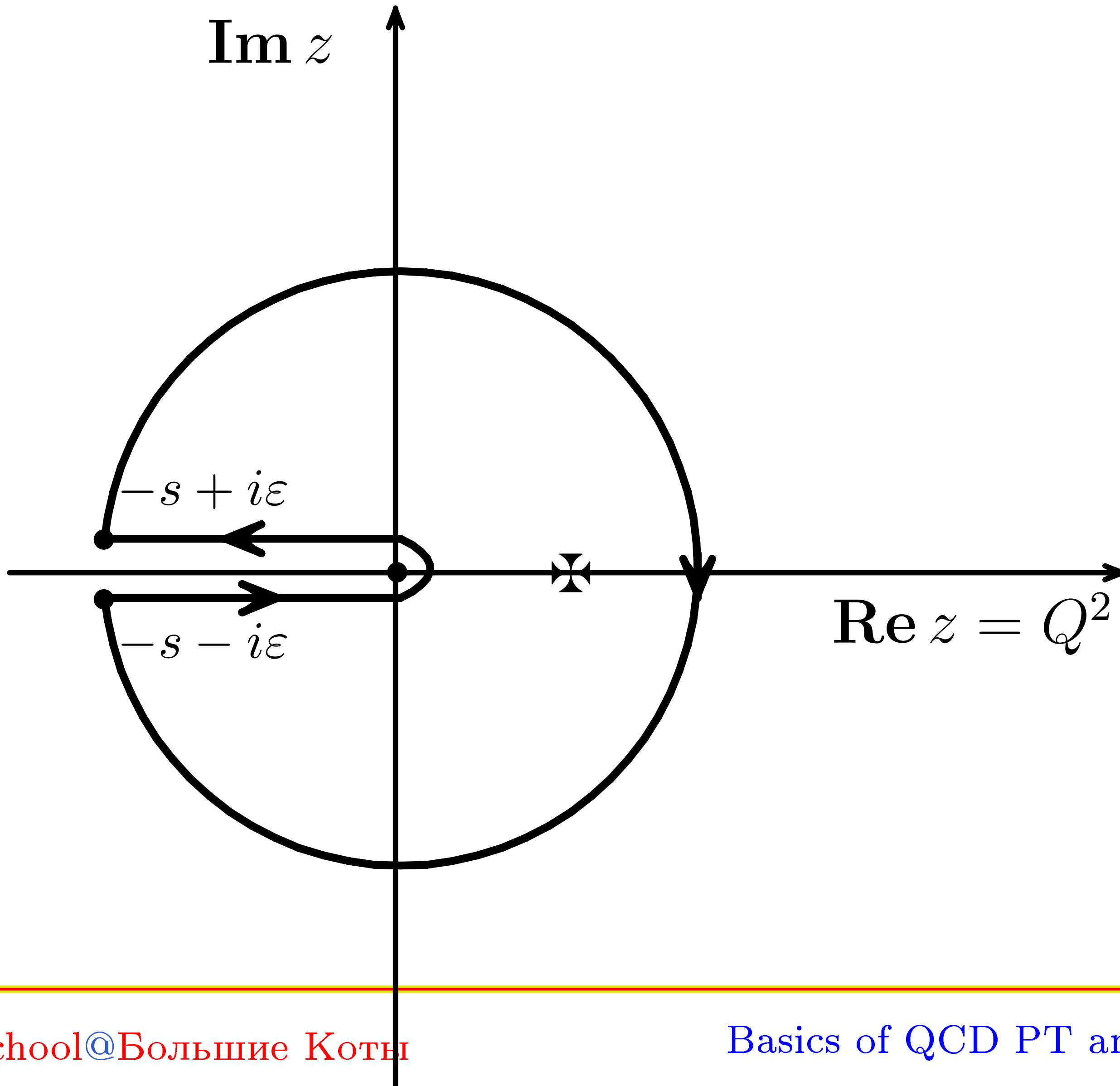
Problem in QCD PT: Minkowski region?

This change of integration contour is legitimate if $D(z)f(z)$ is analytic inside



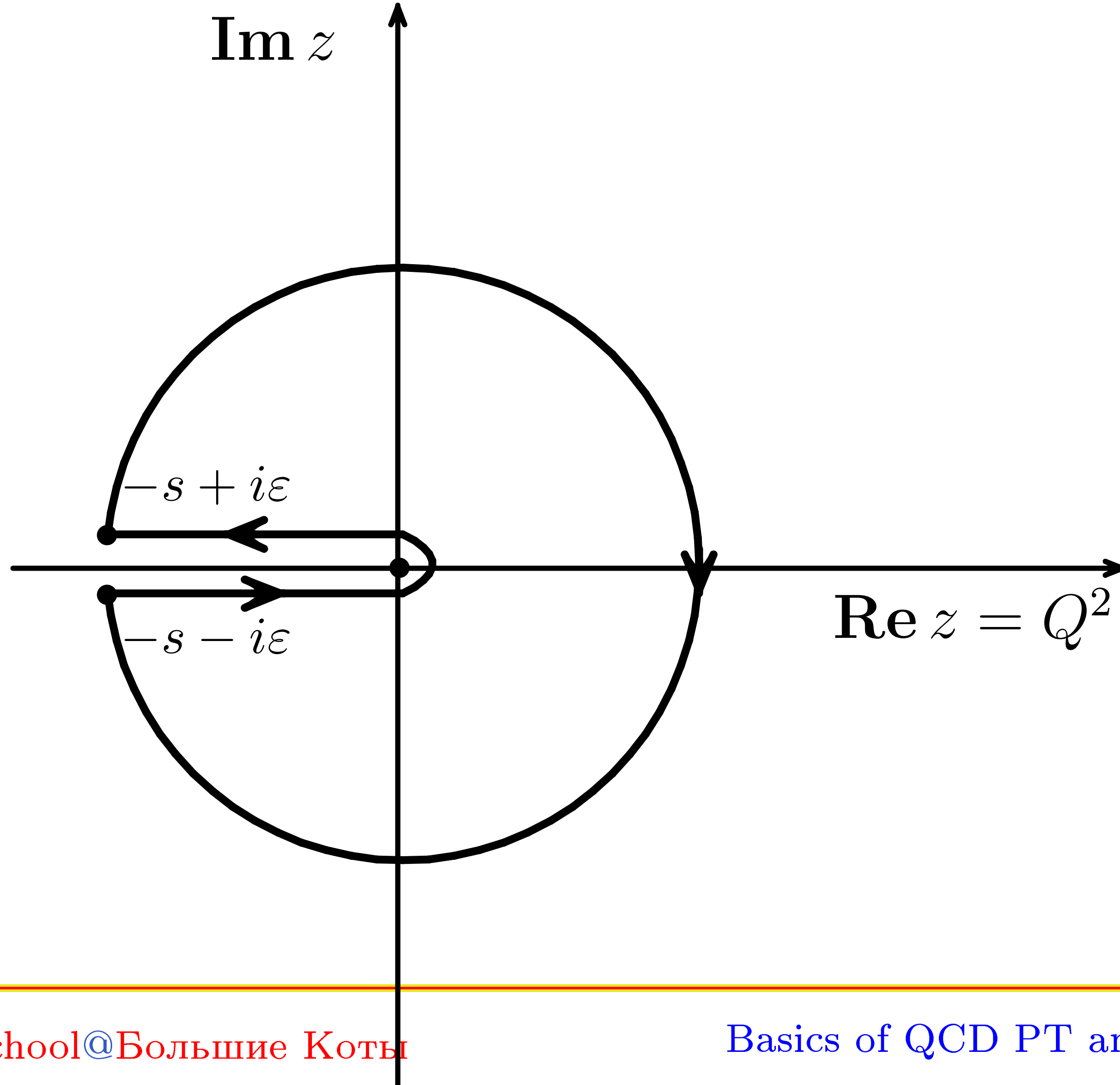
Problem in QCD PT: Minkowski region?

But $\alpha_s(z)$ and hence $D(z)f(z)$ have Landau pole singularity just inside!



Problem in QCD PT: Minkowski region?

In **APT** effective couplings $\mathcal{A}_n(z)$ are analytic functions $\Rightarrow$ this problem does not appear!



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- Defended candidate (1979) and doctorate theses (2000) in JINR (Dubna).
- Creation and elaboration of APT in QCD: 1996–2007. Basic publication in **Phys. Rev. Lett. (1997)** collected about 500 citations.

Basics of Analytic PT

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- PT $\sum_m d_m a_s^m(Q^2) \Rightarrow \sum_m d_m \mathcal{A}_m(Q^2)$ APT

  m – power $\Rightarrow$ m – index

  Here d_m are *numbers* in $\overline{\text{MS}}$ -scheme

Spectral representation

By **analytization** we mean “Källen–Lehman” representation

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma$$

with spectral density $\rho_f(\sigma) = \mathbf{Im}\left[f(-\sigma)\right]/\pi$.

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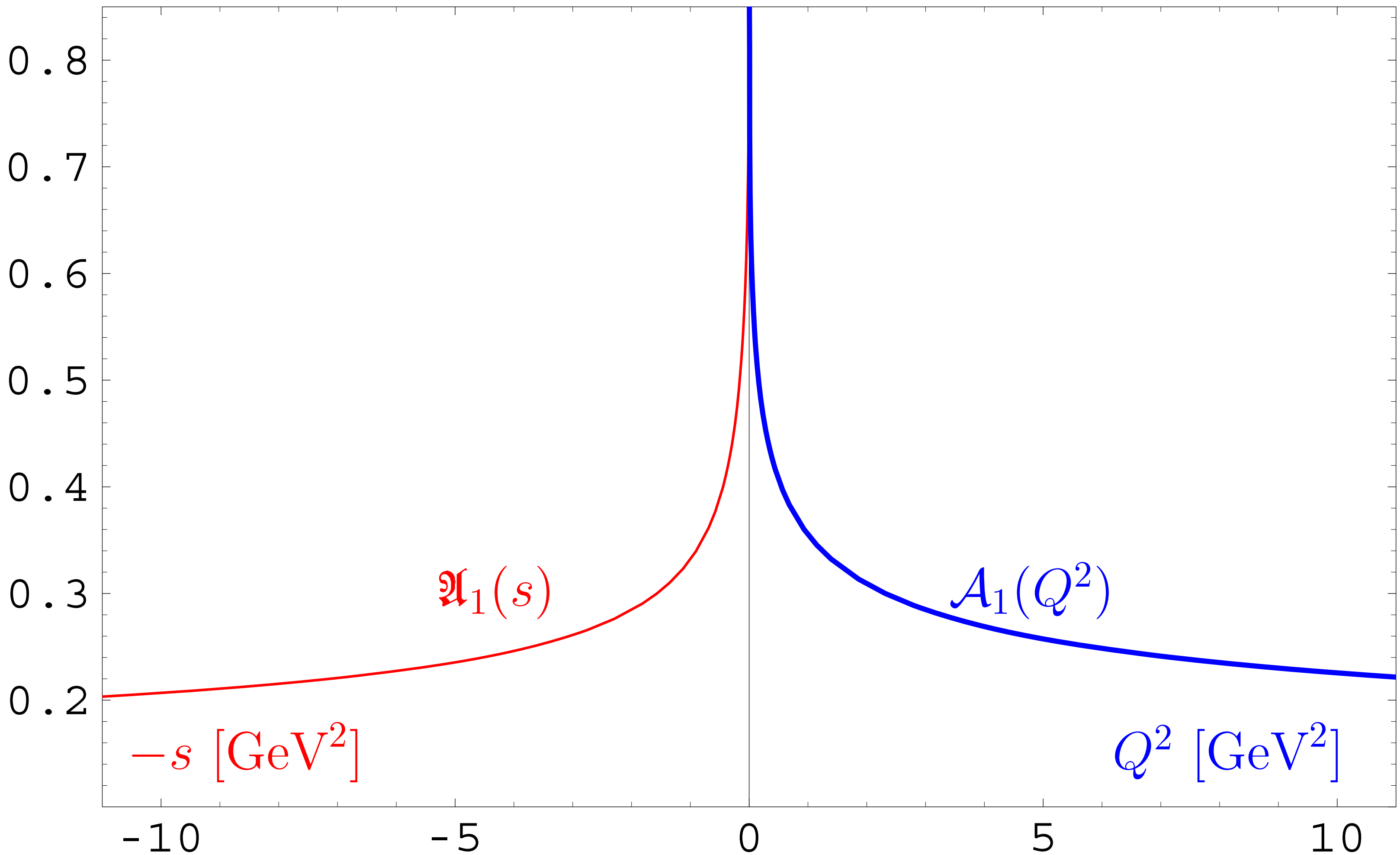
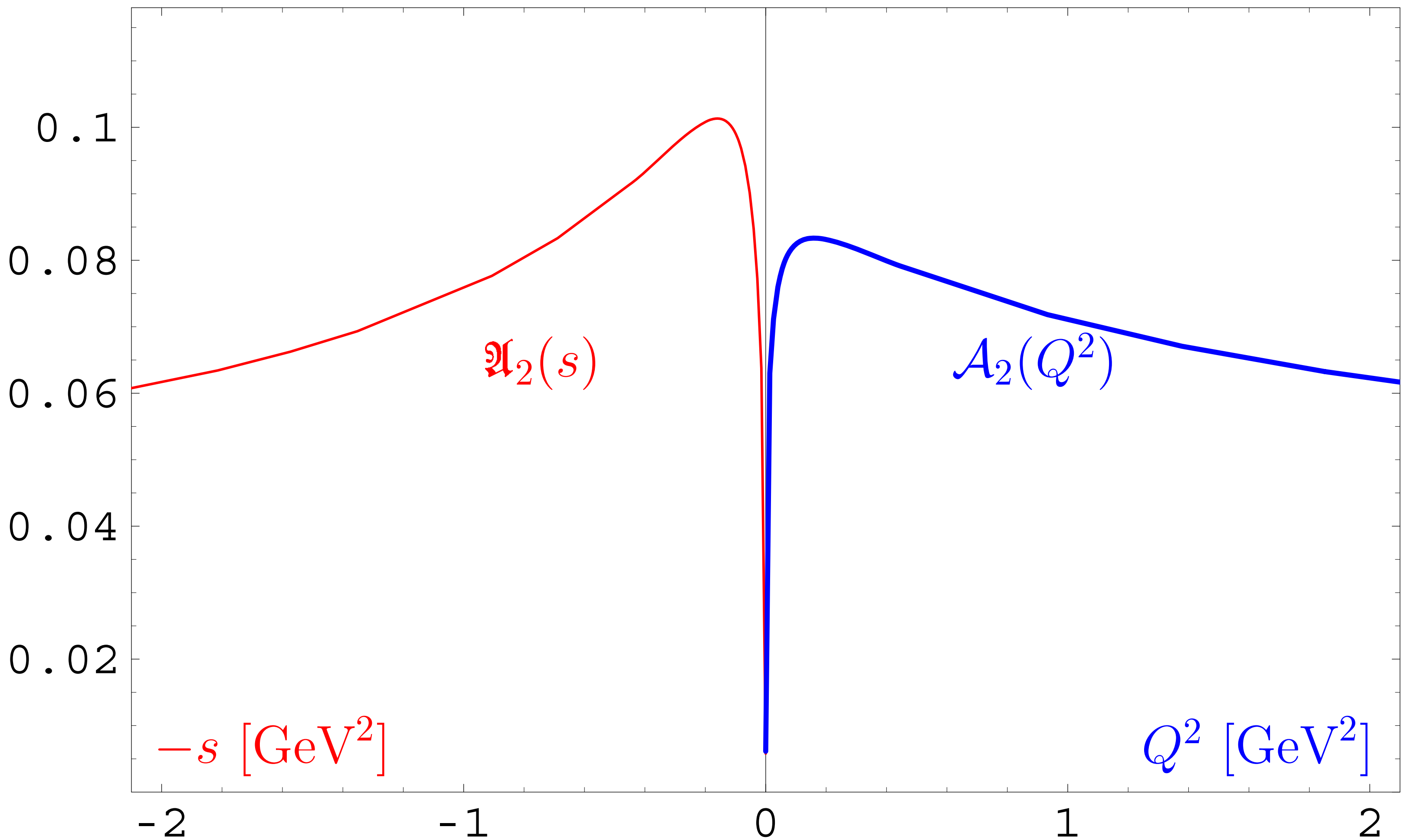
APT graphics: Distorting mirror

First, couplings: $\mathfrak{A}_1(s)$ and $\mathcal{A}_1(Q^2)$



APT graphics: Distorting mirror

Second, square-images: $\mathfrak{A}_2(s)$ and $\mathcal{A}_2(Q^2)$



*Comparison of **PT**, **APT**, and **FAPT***

Theory	PT	APT	FAPT
Set	$\left\{a^{\nu}\right\}_{\nu\in\mathbb{R}}$	$\left\{\mathcal{A}_m, \mathfrak{A}_m\right\}_{m\in\mathbb{N}}$	$\left\{\mathcal{A}_\nu, \mathfrak{A}_\nu\right\}_{\nu\in\mathbb{R}}$
Series	$\sum_m f_m\, a^m$	$\sum_m f_m\, \mathcal{A}_m$	$\sum_m f_m\, \mathcal{A}_m$
Inv. powers	$(a[L])^{-m}$	—	$\mathcal{A}_{-m}[L] = L^m$
Products	$a^\mu a^\nu = a^{\mu+\nu}$	—	—
Index deriv.	$a^\nu \ln^k a$	—	$\mathcal{D}^k \mathcal{A}_\nu$
Logarithms	$a^\nu L^k$	—	$\mathcal{A}_{\nu-k}$

Resummation in one-loop APT

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Consider series $$\mathcal{D}[L] = d_0 + d_1 \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \, \mathcal{A}_n[L] \, .$$

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Consider series $\mathcal{D}[L] = d_0 + d_1 \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \mathcal{A}_n[L]\,.$

Let exist the generating function $P(t)$ for coefficients:

$$d_n = d_1 \int_0^\infty P(t)\, t^{n-1} dt \quad \text{with} \quad \int_0^\infty P(t)\, dt = 1\,.$$

We define a shorthand notation

$$\langle\langle f(t) \rangle\rangle_{P(t)} \equiv \int_0^\infty f(t)\, P(t)\, dt\,.$$

Then coefficients $d_n = d_1 \langle\langle t^{n-1} \rangle\rangle_{P(t)}$.

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Consider series $\mathcal{D}[L] = d_0 + d_1 \sum_{n=0}^{\infty} \langle\langle t^n \rangle\rangle_{P(t)} \mathcal{A}_{n+1}[L]\,.$

We have one-loop recurrence relation:

$$\mathcal{A}_{n+1}[L] = \frac{1}{n!}\left(-\frac{d}{dL}\right)^n \mathcal{A}_1[L]\,.$$

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Result:

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Result:

$$\mathcal{D}[L] = d_0 + d_1 \langle\langle \mathcal{A}_1[L-t] \rangle\rangle_{P(t)}$$

and for Minkowski region:

$$\mathcal{R}[L] = d_0 + d_1 \langle\langle \mathfrak{A}_1[L-t] \rangle\rangle_{P(t)}$$

Resummation in Global Minkowskian APT

Consider series $\mathcal{R}[L] = d_0 + d_1 \sum_{n=1}^{\infty} \langle\langle t^{n-1} \rangle\rangle_{P(t)} \, \mathfrak{A}_n^{\textsf{glob}}[L]$

Result:

$$\mathcal{R}[L] = d_0 \; + \; d_1 \langle\langle \theta\left(L < L_4\right) \left[\Delta_4 \overline{\mathfrak{A}}_1[t] + \overline{\mathfrak{A}}_1\left[L - \frac{t}{\beta_3}; 3\right]\right] \rangle\rangle_{P(t)}$$

$$+ \; d_1 \langle\langle \theta\left(L \geq L_4\right) \overline{\mathfrak{A}}_1\left[L + \lambda_4 - \frac{t}{\beta_4}; 4\right] \rangle\rangle_{P(t)} \, .$$

where

$$\Delta_4 \overline{\mathfrak{A}}_1[t] = \overline{\mathfrak{A}}_1\left[L_4 + \lambda_4 - \frac{t}{\beta_4}; 4\right] - \overline{\mathfrak{A}}_1\left[L_3 - \frac{t}{\beta_3}; 3\right] .$$

Resummation in Global Euclidean APT

In Euclidean domain the result is more complicated:

$$\mathcal{D}[L] = d_0 + d_1 \langle\langle \int_{-\infty}^{L_4} \frac{\overline{\rho}_1\,[L_\sigma; 3]\; dL_\sigma}{1 + e^{L - L_\sigma - t/\beta_3}} \rangle\rangle_{P(t)}$$

$$+ \langle\langle \Delta_4[L, t] \rangle\rangle_{P(t)} + d_1 \langle\langle \int_{L_4}^{\infty} \frac{\overline{\rho}_1\,[L_\sigma + \lambda_4; 4]\; dL_\sigma}{1 + e^{L - L_\sigma - t/\beta_4}} \rangle\rangle_{P(t)}\,.$$

where

$$\Delta_4[L, t] = \int_0^1 \frac{\overline{\rho}_1\,[L_4 + \lambda_4 - tx/\beta_4; 4]\; t}{\beta_4\,\left[1 + e^{L - L_4 - t\bar{x}/\beta_4}\right]}\, dx$$

$$- \int_0^1 \frac{\overline{\rho}_1\,[L_3 - tx/\beta_3; 3]\; t}{\beta_3\,\left[1 + e^{L - L_4 - t\bar{x}/\beta_3}\right]}\, dx.$$

Higgs boson decay

$$H^0 \to b\bar{b}$$

Higgs boson decay into $b\bar{b}$-pair

This decay can be expressed in QCD by means of the correlator of quark scalar currents $J_{\mathsf{S}}(x) = :\bar{b}(x)b(x):$

$$\Pi(Q^2) = (4\pi)^2 i \int dx e^{iqx} \langle 0|\, T[\, J_{\mathsf{S}}(x) J_{\mathsf{S}}(0)\,]\, |0\rangle$$

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in terms of discontinuity of its imaginary part

$$R_{\mathsf{S}}(s) = \mathbf{Im}\,\Pi(-s - i\epsilon)/(2\pi\, s)\,,$$

so that

$$\Gamma(\mathbf{H} \to b\bar{b}) = \frac{G_F}{4\sqrt{2}\pi} M_{\mathbf{H}}\, m_b^2(M_{\mathbf{H}})\, R_{\mathsf{S}}(s = M_{\mathbf{H}}^2)\,.$$

FAPT(M) analysis of R_S

Running mass $m(Q^2)$ is described by the RG equation

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with RG-invariant mass $\hat{m}^2$ (for b-quark $\hat{m}_b \approx 14.6$ **GeV**) and $\nu_0 = 1.04$, $\nu_1 = 1.86$.

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Model for perturbative coefficients

Let us have a look to coefficients of our series, $\tilde{d}_m = d_m / d_1$, with $d_1 = 17/3$.

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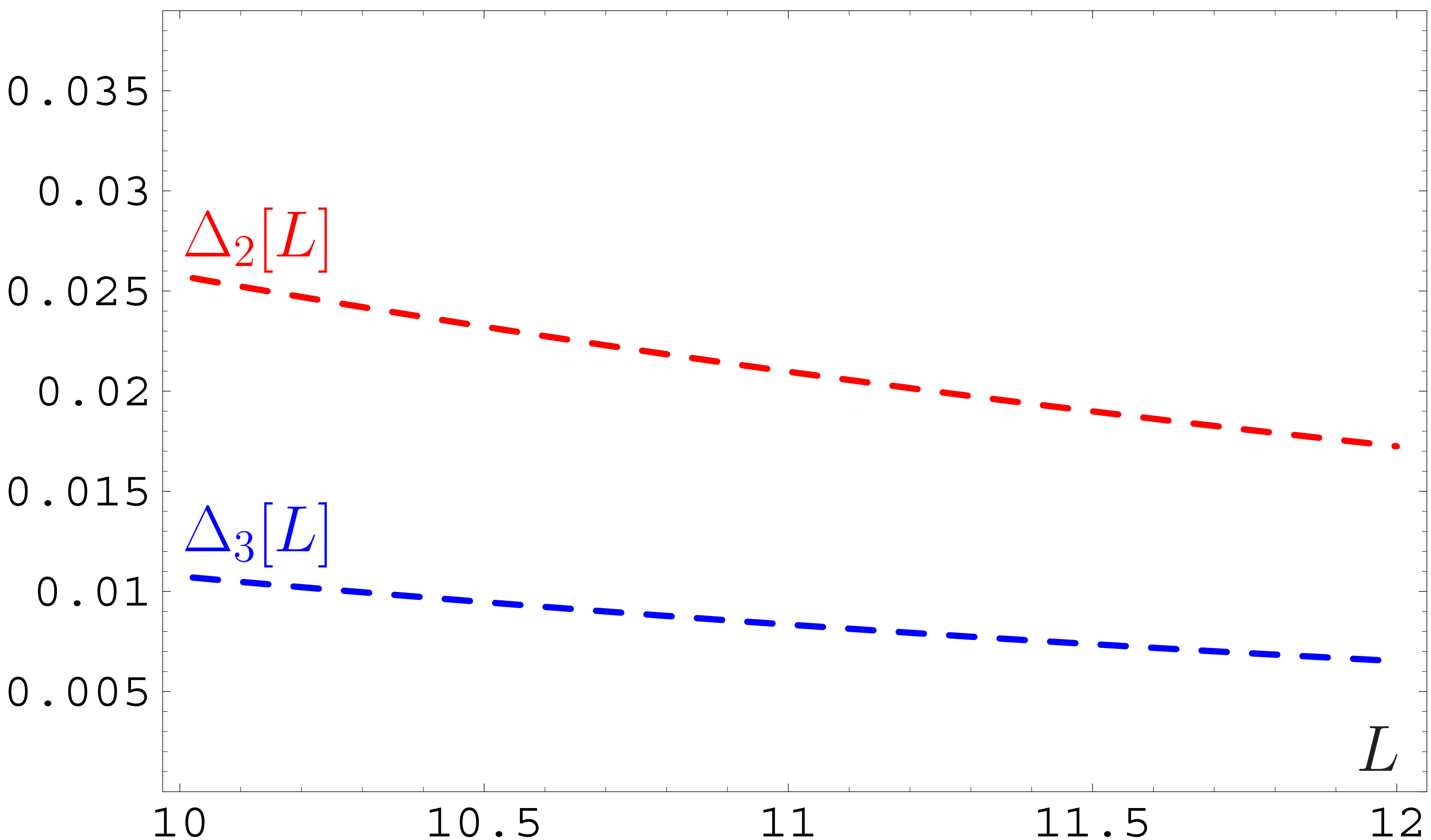
FAPT(M) for $\widetilde{R}_S$: Truncation errors

We define relative errors of series truncation at Nth term:

$$\Delta_N[L] = 1 - \widetilde{\mathcal{R}}_{\mathbf{S}}^{(1;N)}[L] / \widetilde{\mathcal{R}}_{\mathbf{S}}^{(1;\infty)}[L]$$



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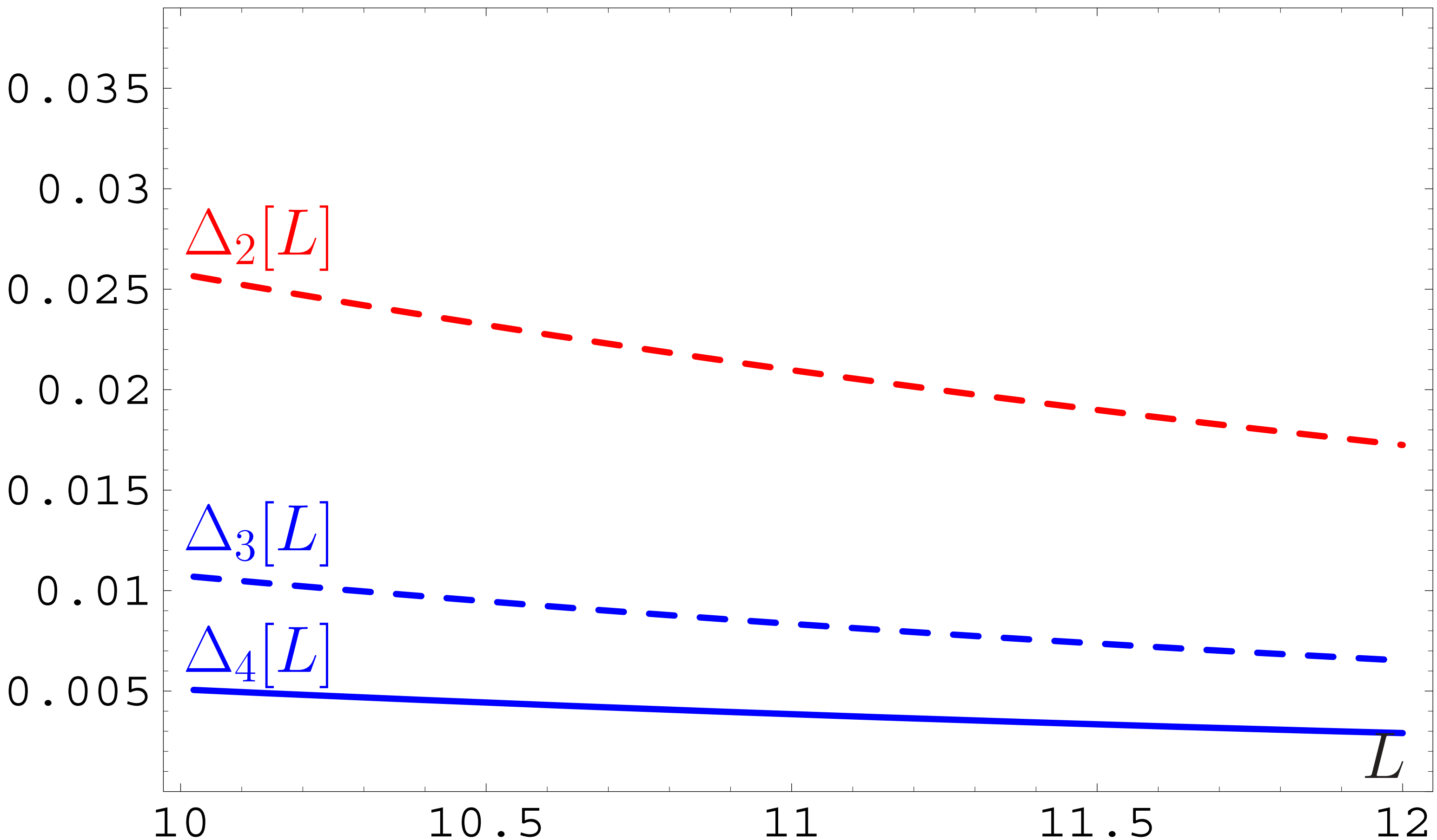

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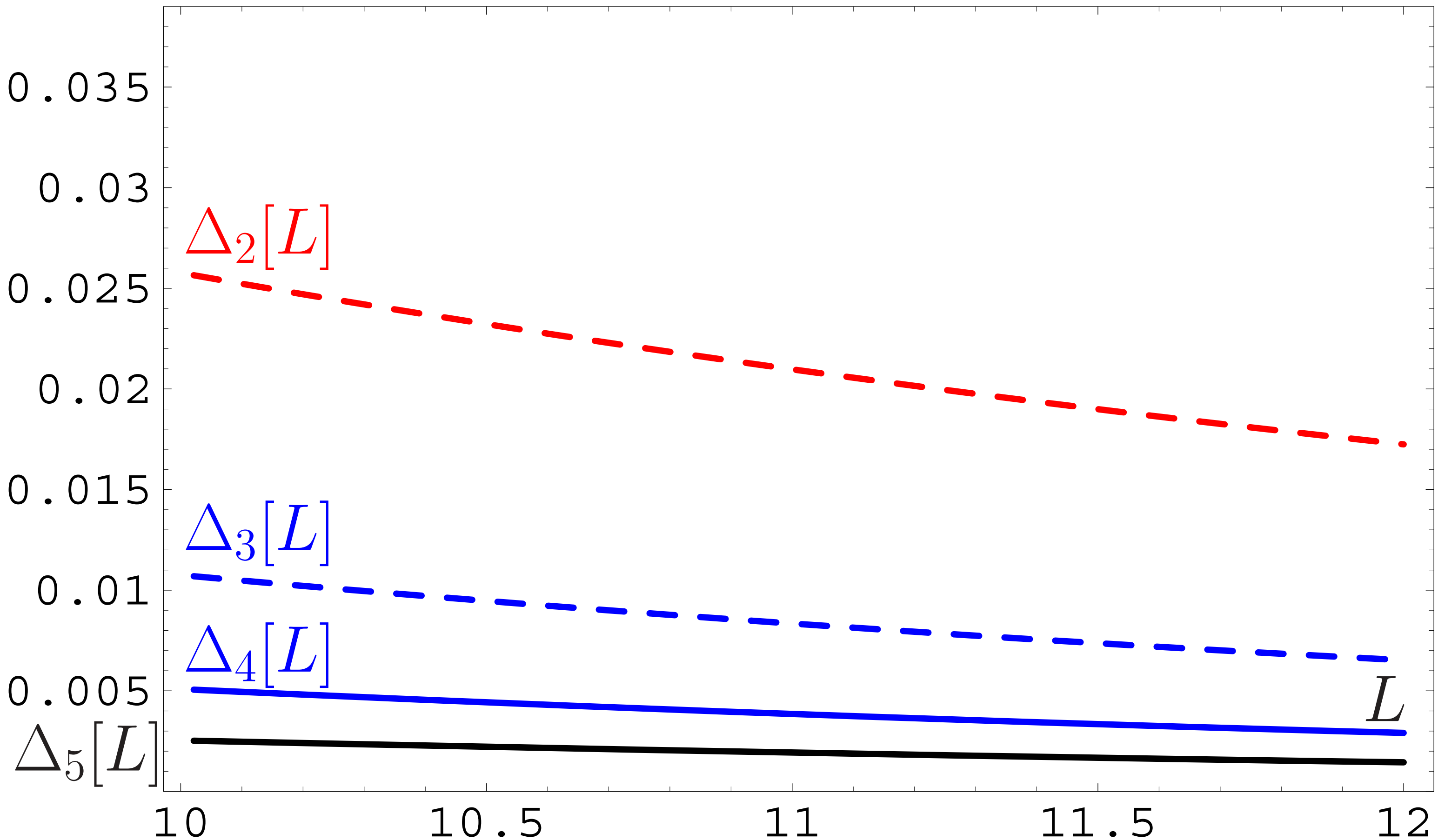

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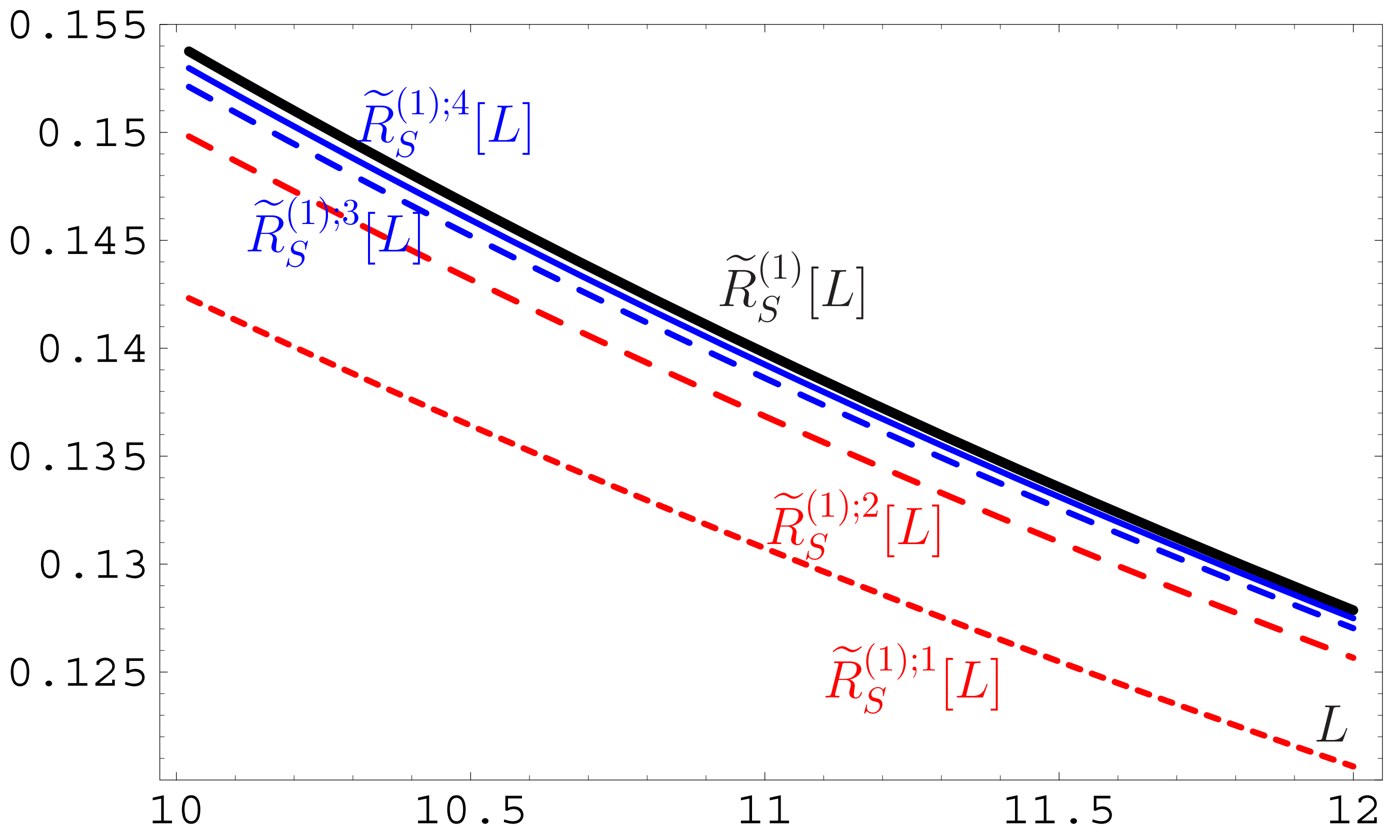
But profit will be tiny — instead of $\mathbf{0.5\%}$ one'll obtain $\mathbf{0.3\%}$!



Adler function $D(Q^2)$ and ratio $R(s)$

Adler function $D(Q^2)$ in vector channel

Adler function $D(Q^2)$ can be expressed in QCD by means of the correlator of quark vector currents

$$\Pi_{\mathsf{V}}(Q^2) = \frac{(4\pi)^2}{3q^2}\, i \int dx\, e^{iqx} \langle 0|\, T[\, J_\mu(x) J^\mu(0)\,]\, |0\rangle$$

in terms of discontinuity of its imaginary part

$$R_{\mathsf{V}}(s) = \frac{1}{\pi}\,\mathrm{Im}\,\Pi_{\mathsf{V}}(-s - i\epsilon)\,,$$

so that

$$D(Q^2) = Q^2 \int_0^\infty \frac{R_{\mathsf{V}}(\sigma)}{(\sigma + Q^2)^2}\, d\sigma\,.$$

APT analysis of $D(Q^2)$ *and* $R_V(s)$

QCD PT gives us

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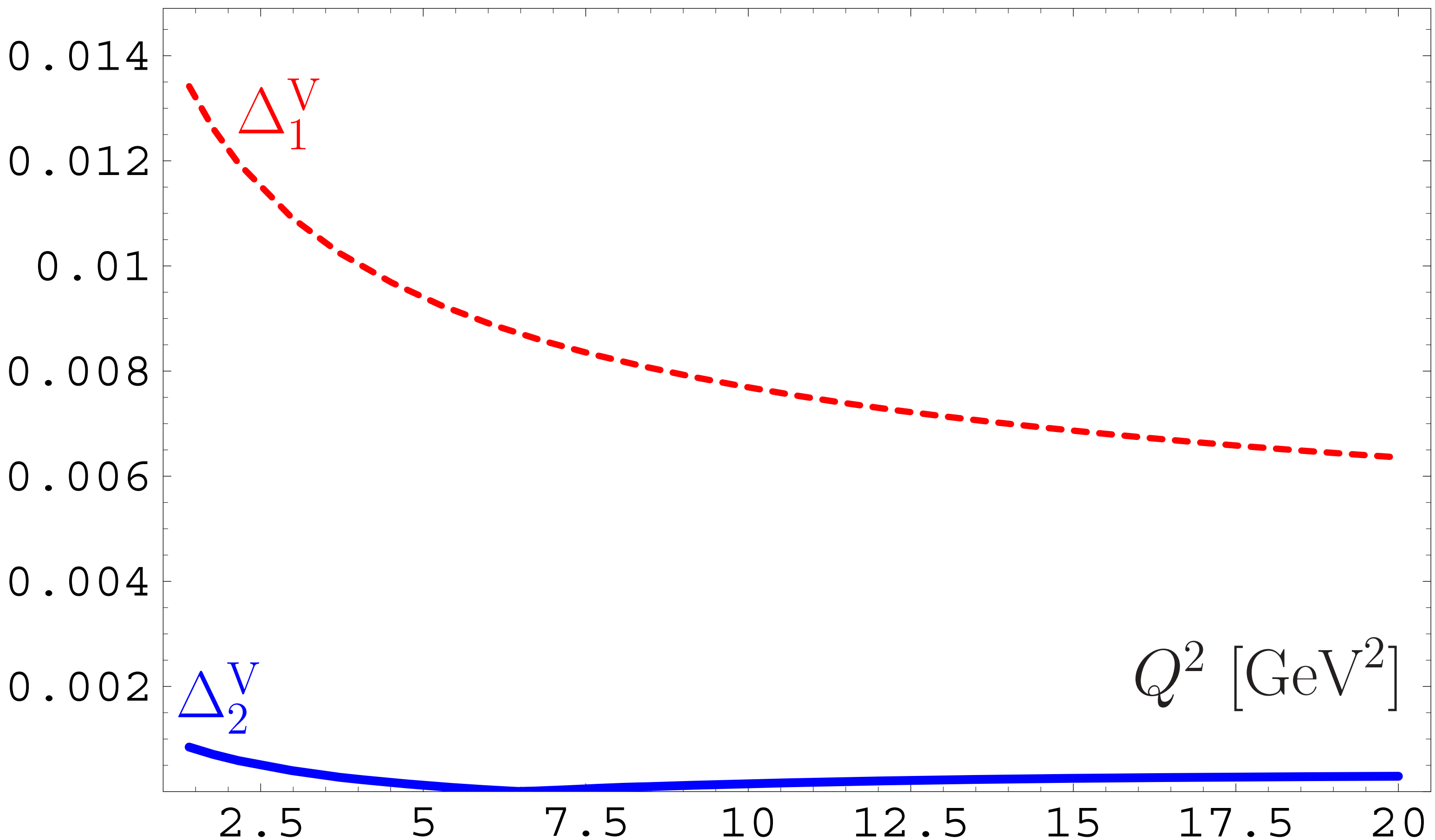

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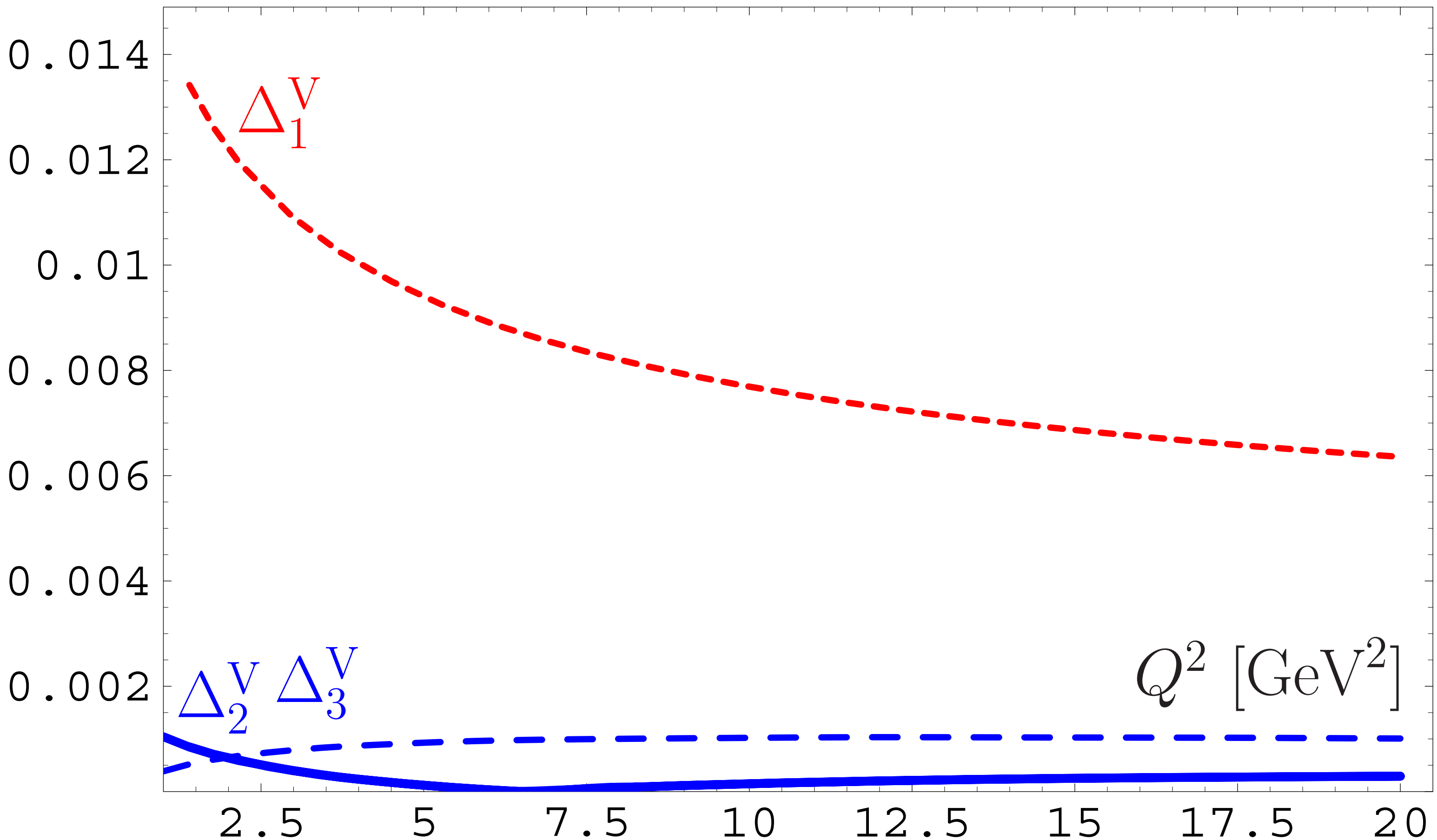
Conclusion: The best accuracy (better than 0.1%) is achieved for **N^2LO** approximation.



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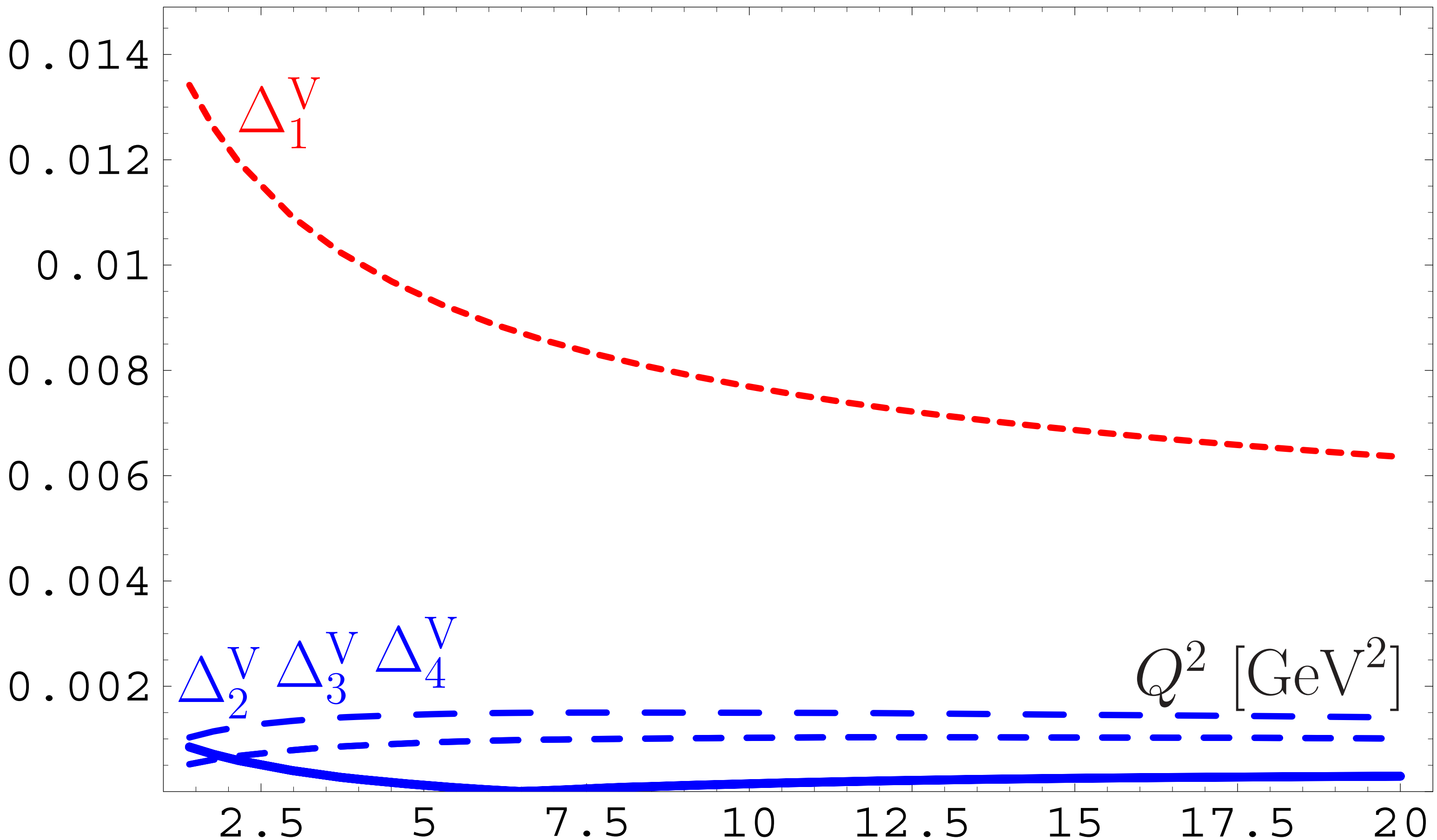
Conclusion: If we add more terms **N^{3}LO** — truncation error increases.



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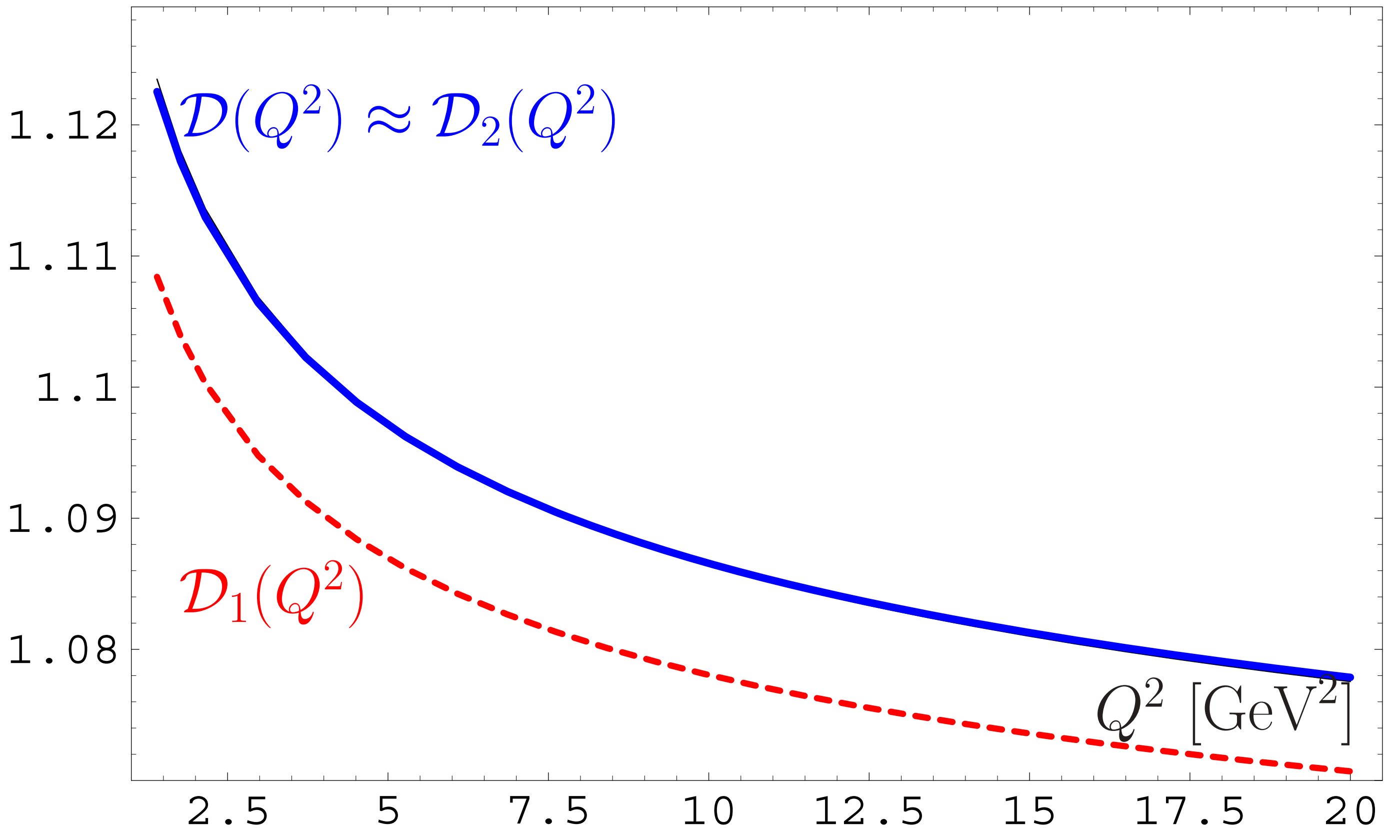

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Do not calculate higher-order corrections!
Use instead APT and FAPT!