Lecture 1: Sum Rules in Quantum Mechanics

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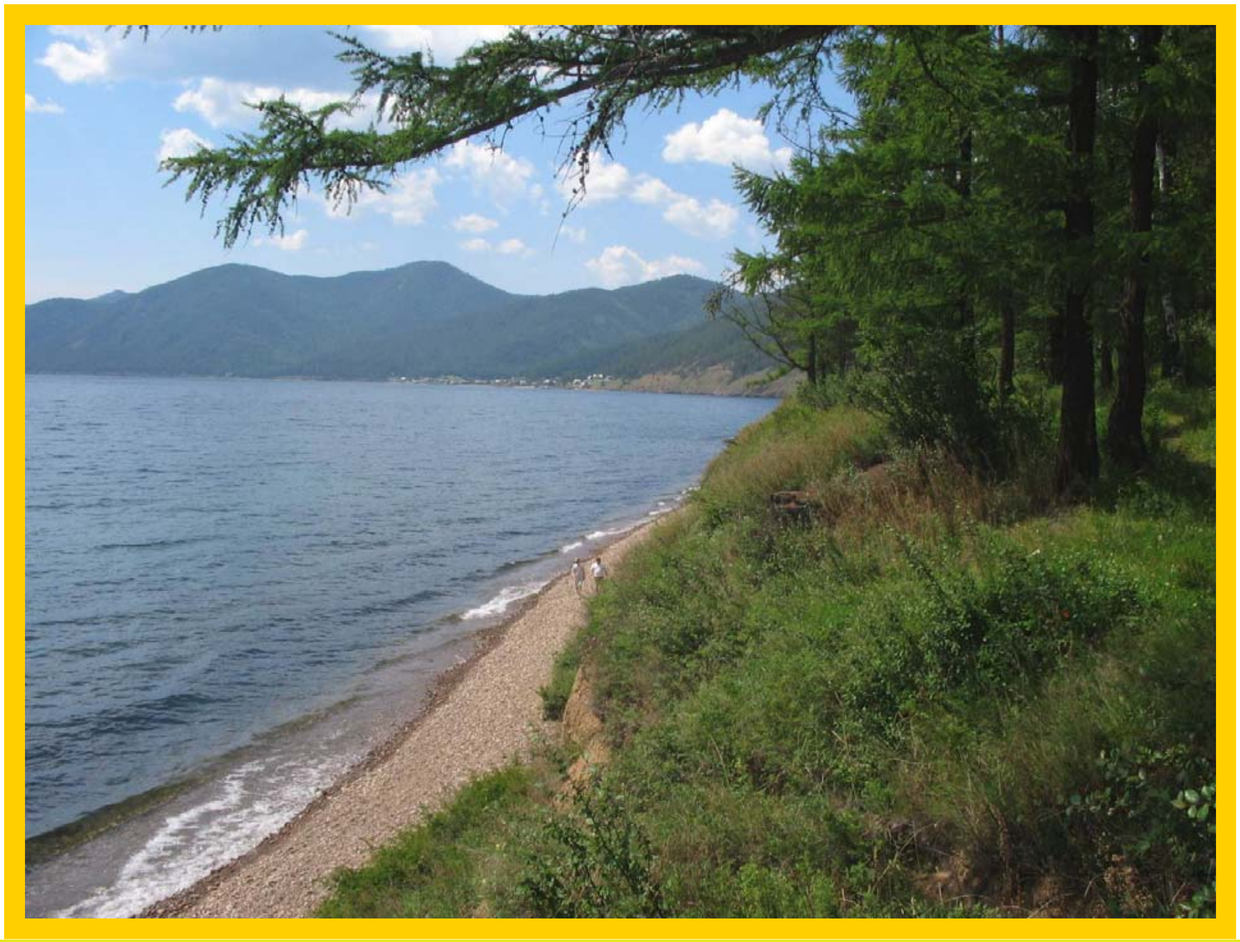

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- QCD: Currents, Correlators and Spectral Densities of Real Particles.

Quantum-mechanical toy model:

Two-Dimensional

Harmonic

Oscillator

Two-Dimensional Oscillator

Simplest system with confinement – oscillator with potential $V(\vec{r}) = m\omega^2 r^2/2$. All formulas greatly simplify if $D = 2$.

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$$E_n = (?????)\omega \, ,$$

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We will consider the regular quasi-perturbative method of Sum Rules to determine energy E_0 and $|\psi_0(0)|^2$ of the ground state.

General scheme of Sum Rule method

The general scheme of Sum Rule method

- We study correlator $M(\mu)$, which has spectral expansion:

$$M^{\textsf{spec}}(\mu) = |\psi_0(0)|^2 \, e^{-E_0/\mu} + \textbf{“higher states”}$$

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- We construct perturbative expansion of this correlator:

$$M^{\text{pert}}(\mu) = M_0(\mu) + \sum_{n\geq 1} C_{2n}\frac{\omega^{2n}}{\mu^{2n}}\,,$$

where $M_0(\mu)$ corresponds to free particle and has dispersion representation:

$$M_0(\mu) = \int_0^\infty \rho_0(E)\, e^{-E/\mu}\, dE\,.$$

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- We construct perturbative expansion of this correlator:

$$M^{\text{pert}}(\mu) = M_0(\mu) + \sum_{n\geq 1} C_{2n} \frac{\omega^{2n}}{\mu^{2n}}$$

- Sum Rule – it is simply

$$\boxed{M^{\text{spec}}(\mu) = M^{\text{pert}}(\mu)}$$

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- Our aim: to determine $|\psi_0(0)|^2$ and E_0 from this SR by calculating spectral density $\rho_0(E)$ and coefficients C_{2n} and by demanding stability of this SR in variable $\mu \in [\mu_{\mathsf{L}}, \mu_{\mathsf{U}}]$.

Green functions and Correlators

$M(\mu)$ and Green function $G(\vec{x}, t)$

- Consider 2-time Green function

$$G(0,0|\vec{x}_{\mathsf{f}}, t_{\mathsf{f}}) = \sum_{k\geq 0} \psi_k^*(\vec{x}_{\mathsf{f}})\psi_k(0)e^{-iE_k t_{\mathsf{f}}}$$

$=$ probability amplitude for $(\vec{x}_{\mathsf{in}} = 0, t_{\mathsf{in}} = 0) \longrightarrow (\vec{x}_{\mathsf{f}}, t_{\mathsf{f}})$.

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- To get $M(\mu)$ put $\vec{x}_{\mathsf{f}} = 0$, $t_{\mathsf{f}} = 1/i\mu$:

$$M(\mu) = G(0,0|0,1/i\mu) = \sum_{k\geq 0} \left|\psi_k(0)\right|^2 e^{-E_k/\mu} = M^{\mathsf{spec}}(\mu)$$

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Question: here $k = (n,l)$.
What is about states with $l \neq 0$?

$M(\mu)$ and Green function $G(\vec{x}, t)$

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In our case $\left|\psi_n(0)\right|^2 = m\omega/\pi$, so we have

$$M(\mu) = \ ???$$

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$$M(\mu) = \frac{m\omega}{2\pi\, \sinh\left(\omega/\mu\right)}$$

Spectral expansion for $M(\mu)$

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$$M^{\text{spec}}(\mu) = \frac{m\omega}{\pi}\,\frac{1}{e^{\omega/\mu} - e^{-\omega/\mu}} = ???$$

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$$M(\omega) = \frac{m\omega}{2\pi} \cdot (0.851)$$

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Numerically at $\mu = \omega$:

$$M^{\text{spec}}(\omega) = \frac{m\omega}{2\pi}\,(0.736 + 0.100 + 0.013 + 0.002 + \ldots)$$

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Ground state contributes 86%, first excitation – 12%, while the second – 1.5%.

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$$M^{\mathrm{pert}}(\mu) = \frac{m\mu}{2\pi}\left(1 - \frac{\omega^2}{6\mu^2} + \frac{7}{360}\frac{\omega^4}{\mu^4} - \frac{31}{15120}\frac{\omega^6}{\mu^6} + \dots\right)$$

Here $m\mu/2\pi$ corresponds to Green function of ... ?

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$$M^{\text{free}}(\mu) = \frac{m\mu}{2\pi}$$

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Numerically at $\mu = \omega$:

$$M^{\text{pert}}(\omega) = \frac{m\omega}{2\pi}\,(1 - 0.167 + 0.019 - 0.002 + \ldots)$$

First correction specifies free result by 17%, while the second – by 3%

Asymptotic Freedom for HO Correlator

Asymptotic Freedom for $M(\mu)$

Perturbative expansion can be rewritten

$$\frac{M(\mu) - M_0(\mu)}{M_0(\mu)} = -\frac{\omega^2}{6\mu^2} + \frac{7}{360}\frac{\omega^4}{\mu^4} - \frac{31}{15120}\frac{\omega^6}{\mu^6} + \dots$$

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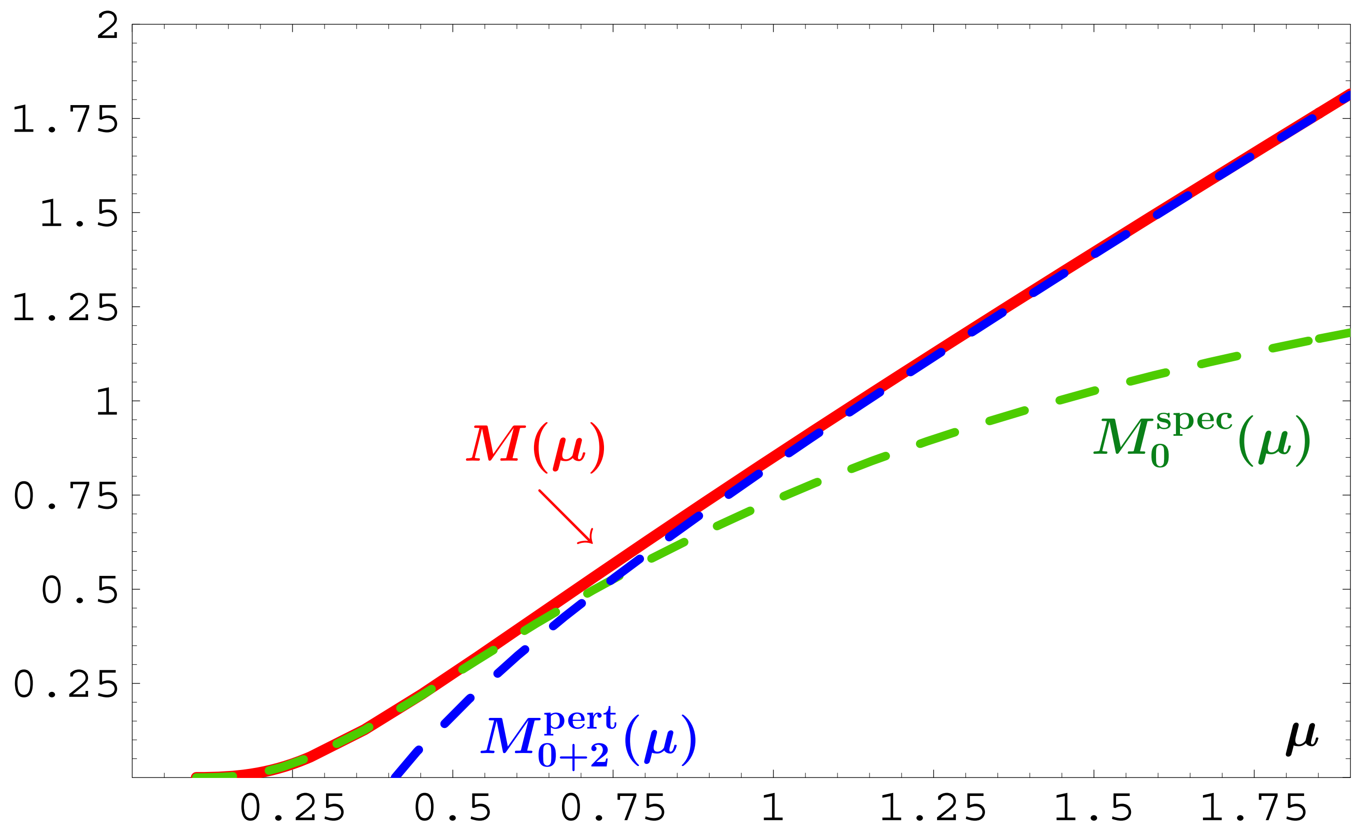
Asymptotic Freedom in Quantum Mechanics

is violated by **Power Corrections**

of the type ω^2/μ^2

Graphics for $M(\mu)$

Exact $M(\mu)$; **Ground state only**; $M_0(\mu) + O(\omega^2/\mu^2)$.



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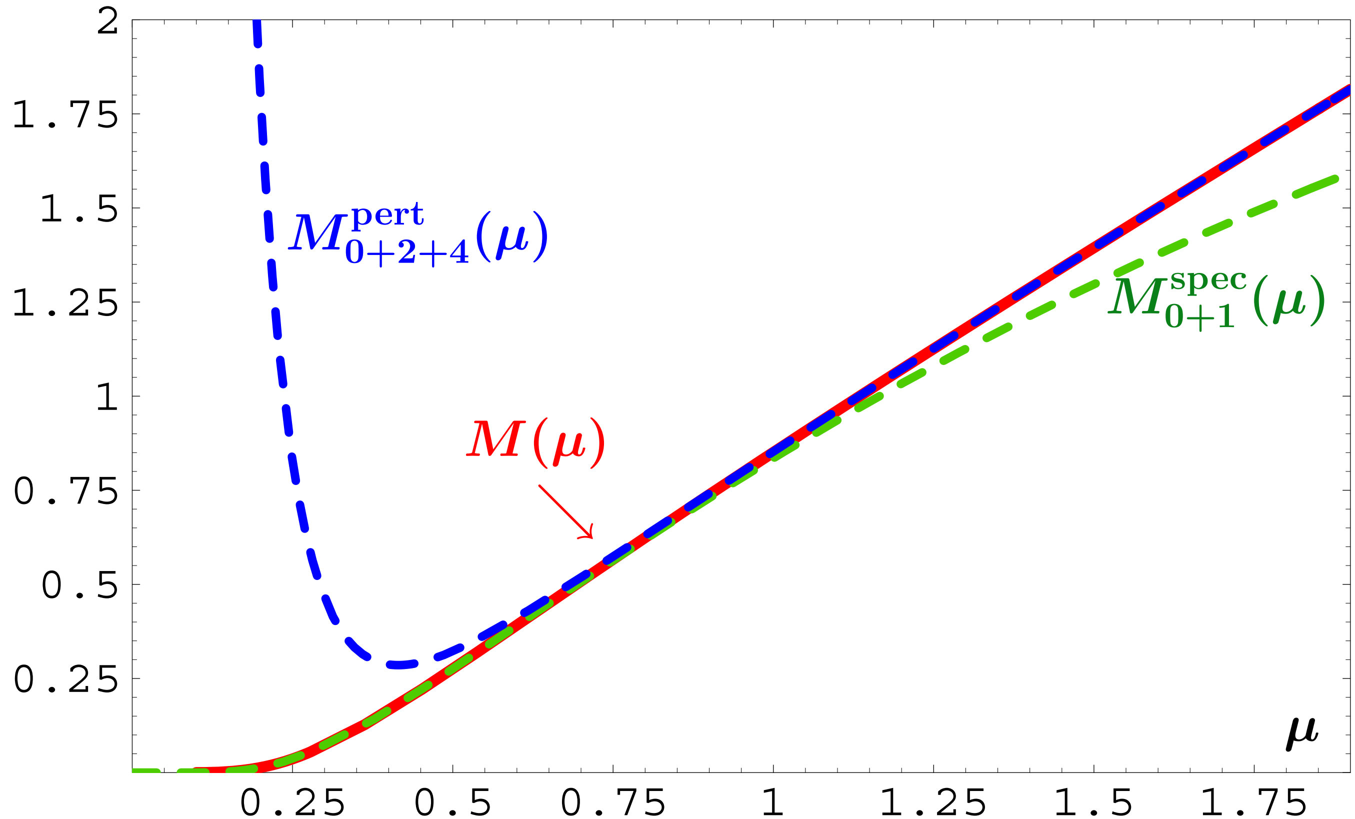
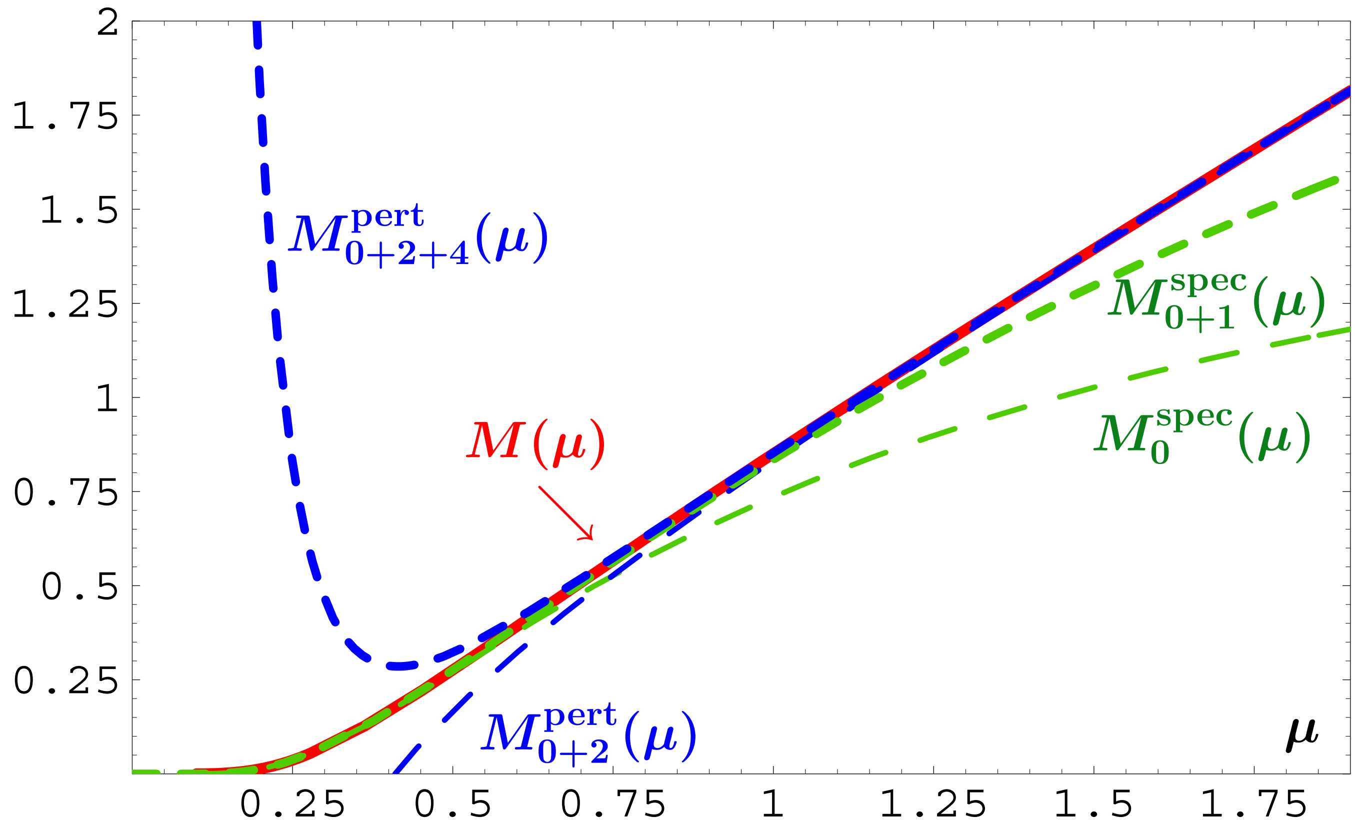


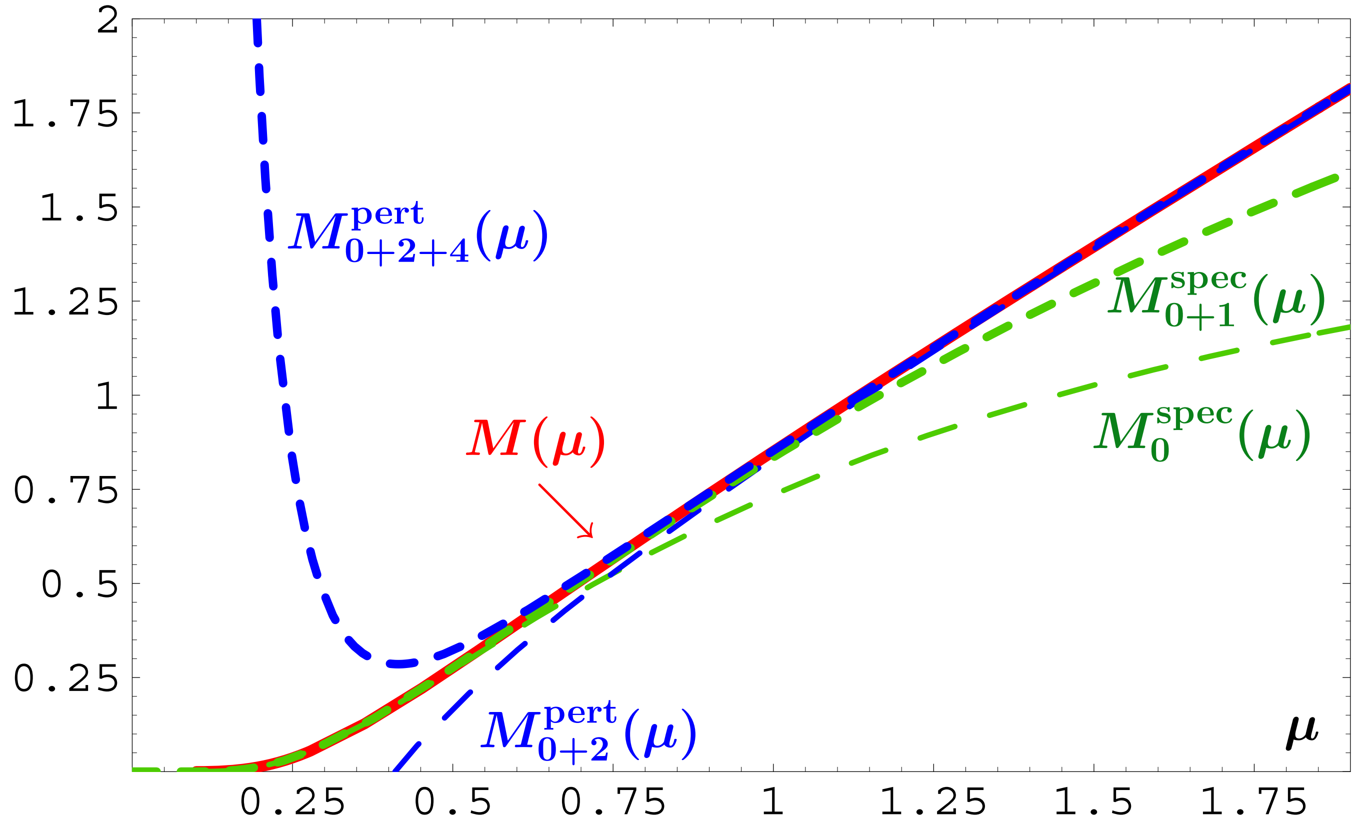
Graphics for $M(\mu)$

For small μ in spectral part survives only ground state $|\psi_0|^2 e^{-E_0/\mu}$. **But**: PT breaks down.



Graphics for $M(\mu)$

For large μ **AF** works well: $M(\mu) \simeq M_0(\mu)$. **But**: We need more and more resonances to saturate $M(\mu)$.



Global and Local Dualities

Global Duality: Free ⇔ Confined

We need to model higher resonances in spectral repr. of our correlator $M(\mu)$:

$$M^{\text{spec}}(\mu) = \sum_{k\geq 0} \frac{m\omega}{\pi}\, e^{-E_k/\mu} \equiv \int_0^\infty \rho^{\text{osc}}(E)\, e^{-E/\mu}\, dE$$

What is here the spectral density:

$$\rho^{\text{osc}}(E) = ???$$

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Here spectral density is just sum of δ-functions:

$$\rho^{\text{osc}}(E) = \sum_{k\geq 0} \frac{m\omega}{\pi} \delta(E - E_k)$$

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Analogously we have integral representation for free correlator:

$$M_0(\mu) = \frac{m\mu}{2\pi} \equiv \int_0^\infty \rho_0(E)\, e^{-E/\mu}\, dE$$

Who knows what is $\rho_0(E)$?

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Who knows what is $\rho_0(E)$? Answer: $\rho_0(E) = \dfrac{m}{2\pi}$.

Global Duality: Free ⇔ Confined

We need to model higher resonances in spectral repr. of our correlator $M(\mu)$:

$$M^{\text{spec}}(\mu) = \int_0^\infty \rho^{\text{osc}}(E)\, e^{-E/\mu}\, dE\,;\ M_0(\mu) = \int_0^\infty \rho_0(E)\, e^{-E/\mu}\, dE$$

Asymptotic Freedom:

$$M(\mu \to \infty) = M_0(\mu \to \infty)$$

Global Duality: Free $\Leftrightarrow$ Confined

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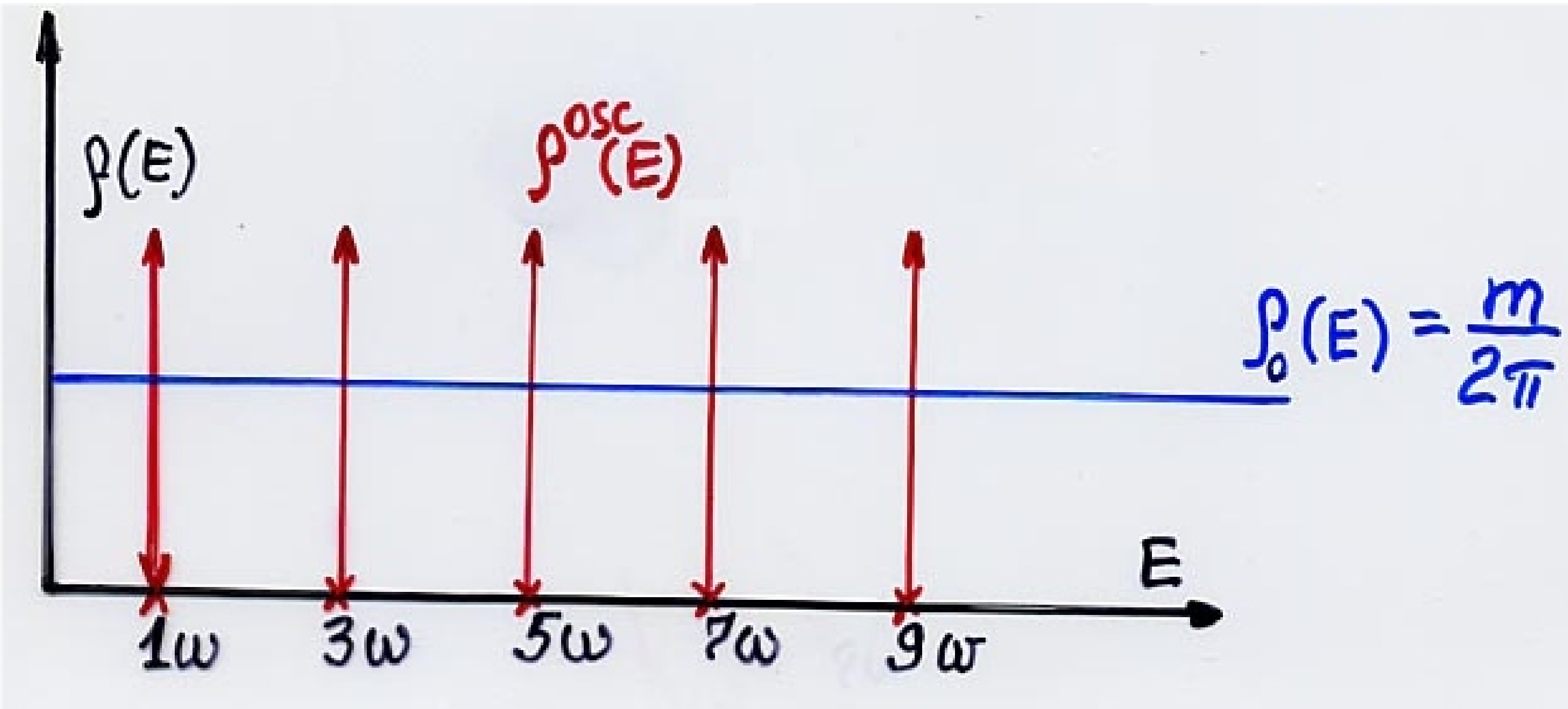
dictates **Global Duality** for these two densities

$$\int_0^\infty \rho^{\text{osc}}(E)\, dE = \int_0^\infty \rho_0(E)\, dE$$

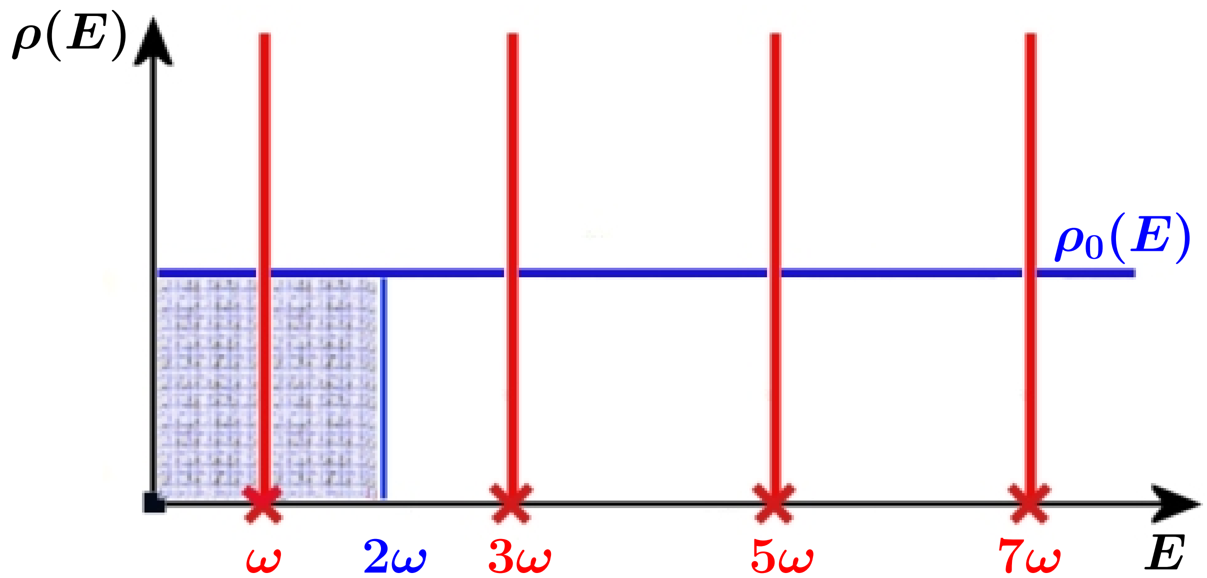
Graphics of dual spectral densities

At first glance they have completely different behaviour:



Graphics of dual spectral densities

But we have very interesting relations between $2k\omega$-partial integral moments of this dual densities, namely, $< E^N >_{2k\omega} = \int_{2k\omega}^{2k\omega+2\omega} E^N \rho(E) dE$.



For $N = 0$:

$$\int_{2k\omega}^{2(k+1)\omega} \rho^{\mathrm{osc}}(E)\, dE = \frac{m\omega}{\pi} = \int_{2k\omega}^{2(k+1)\omega} \rho_0(E)\, dE$$

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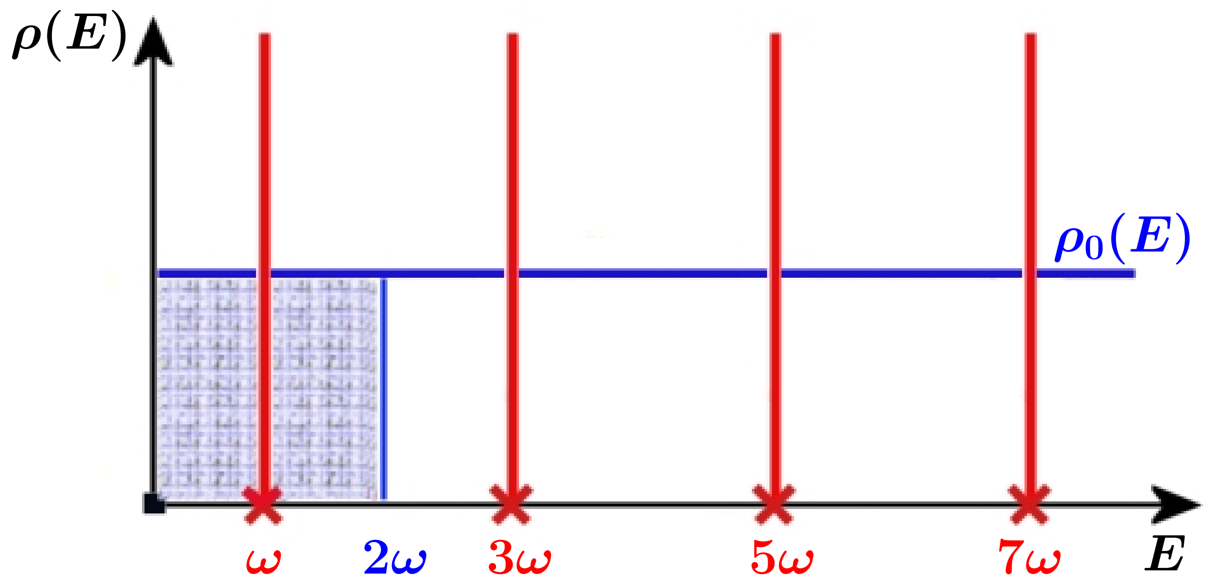

For $N = 1$:

$$\int_{2k\omega}^{2(k+1)\omega} E\,\rho^{\mathrm{osc}}(E)\,dE = \frac{m\omega^2(2k+1)}{\pi} = \int_{2k\omega}^{2(k+1)\omega} E\,\rho_0(E)\,dE$$

Graphics of dual spectral densities

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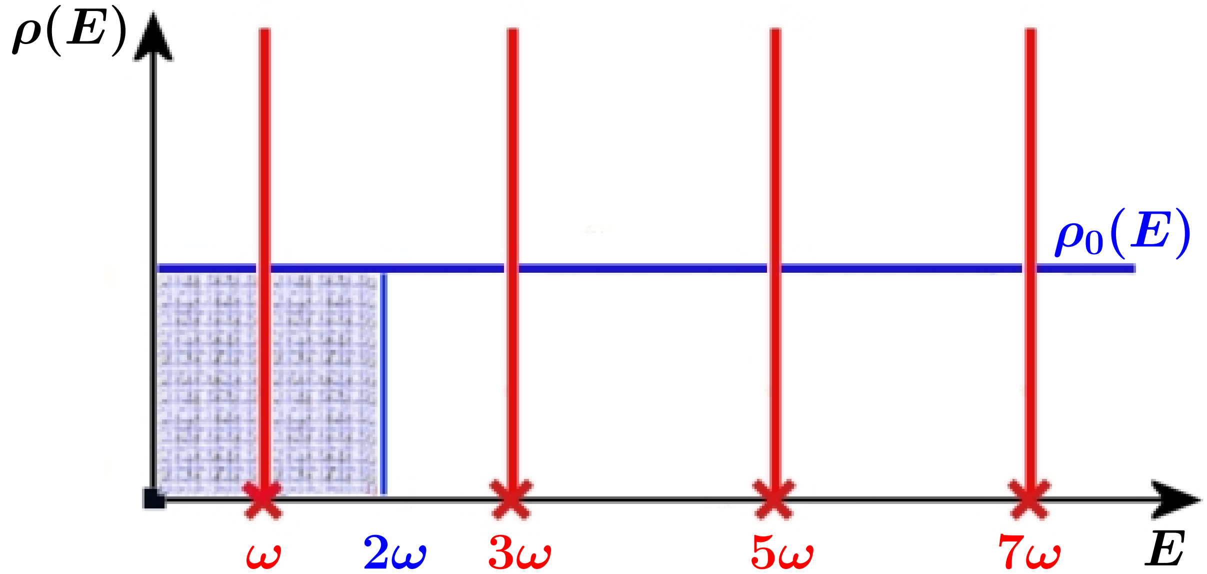
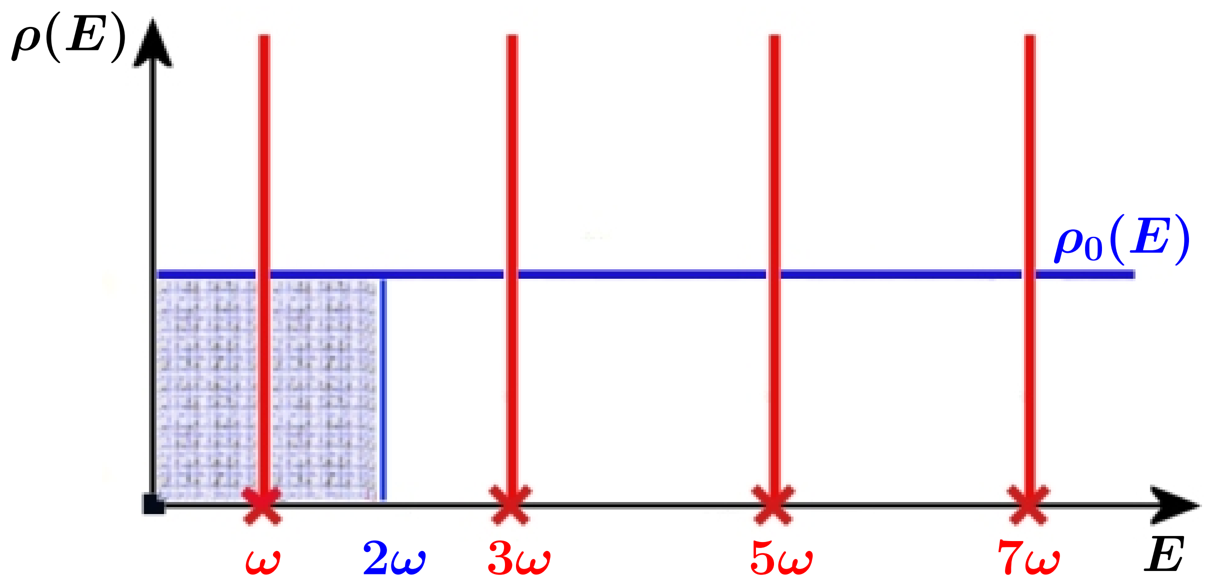


For $N \geq 2$:

$$\int_{2k\omega}^{2(k+1)\omega} E^N \rho^{\text{osc}}(E)\, dE = \int_{2k\omega}^{2(k+1)\omega} E^N \rho_0(E)\, dE \left[1 + O\left(\frac{N^2}{k^2}\right)\right]$$

Graphics of dual spectral densities



We have duality between each excited resonance in oscillator and free particle in some spectral domain $\Rightarrow$ "Local Duality"

QM Sum Rules for Harmonic Oscillator

QM Sum Rules

We can model higher state contributions by

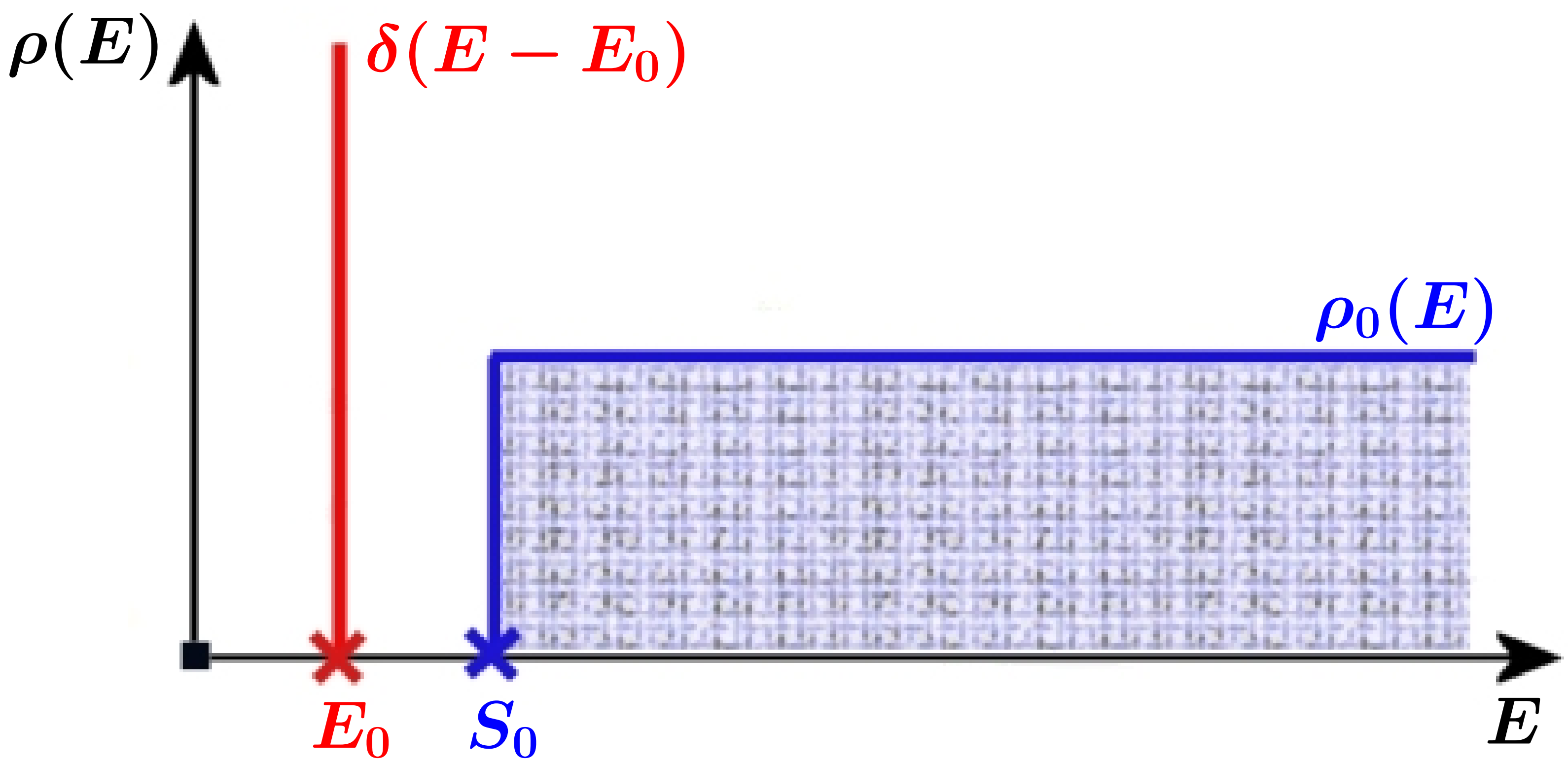
“higher states” $=$ **“free states” outside interval** $(0, S_0)$

QM Sum Rules

We can model higher state contributions by

“higher states” = “free states” outside interval $(0, S_0)$

or: $\rho^{\text{mod}}(E) = |\psi_0(0)|^2\,\delta\,(E - E_0) + \rho_0(E)\,\theta\,(E - S_0)$



QM Sum Rules

Our model for HSs gives

$$M^{\text{mod}}(\mu) = |\psi_0(0)|^2\, e^{-E_0/\mu} + \int_{S_0}^{\infty} \rho_0(s)\, e^{-E/\mu}\, dE$$

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$$M^{\text{mod}}(\mu) = |\psi_0(0)|^2 \, e^{-E_0/\mu} + \int_{S_0}^{\infty} \rho_0(s) \, e^{-E/\mu} \, dE$$

After all we have Sum Rule:

$$|\psi_0(0)|^2 e^{-E_0/\mu} = \int_0^{S_0} \rho_0(E) \, e^{-E/\mu} \, dE + \textbf{power corrections}$$

QM Sum Rules

Our model for HSs gives

$$M^{\text{mod}}(\mu) = |\psi_0(0)|^2\, e^{-E_0/\mu} + \int_{S_0}^{\infty} \rho_0(s)\, e^{-E/\mu}\, dE$$

or equivalent SR (with $\Psi_0(0) \equiv \psi_0(0)\sqrt{\pi/\omega}$):

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Daughter SR — by $\dfrac{-\partial\ldots}{\partial\mu^{-1}}$:

$$|\Psi_0(0)|^2 E_0 e^{-E_0/\mu} = \frac{\mu^2}{2\omega}\left\{1 - \left(1 + \frac{S_0}{\mu}\right) e^{-S_0/\mu} + \frac{\omega^2}{6\mu^2} + \ldots\right\}$$

QM Sum Rules: The Scheme

Main SR:

$$|\Psi_0(0)|^2 \approx \Psi_0^2(E_0, S_0, \mu) = \frac{\mu\, e^{E_0/\mu}}{2\omega}\left\{1 - e^{-S_0/\mu} - \frac{\omega^2}{6\mu^2} + \dots\right\}$$

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QM Sum Rules: Fidelity Window

- Power corrections are of the type $(\omega/\mu)^{2n}$ and they are huge at $\mu \ll \omega$. Demand:

$$\Delta_{\mathsf{pert}}(\mu) \equiv \sum_{n\geq 1} \frac{C_{2n}(\omega/\mu)^{2n}}{M_0(\mu)} \leq 0.33 \quad \text{for all} \quad \mu \geq \mu_{\mathsf{L}}$$

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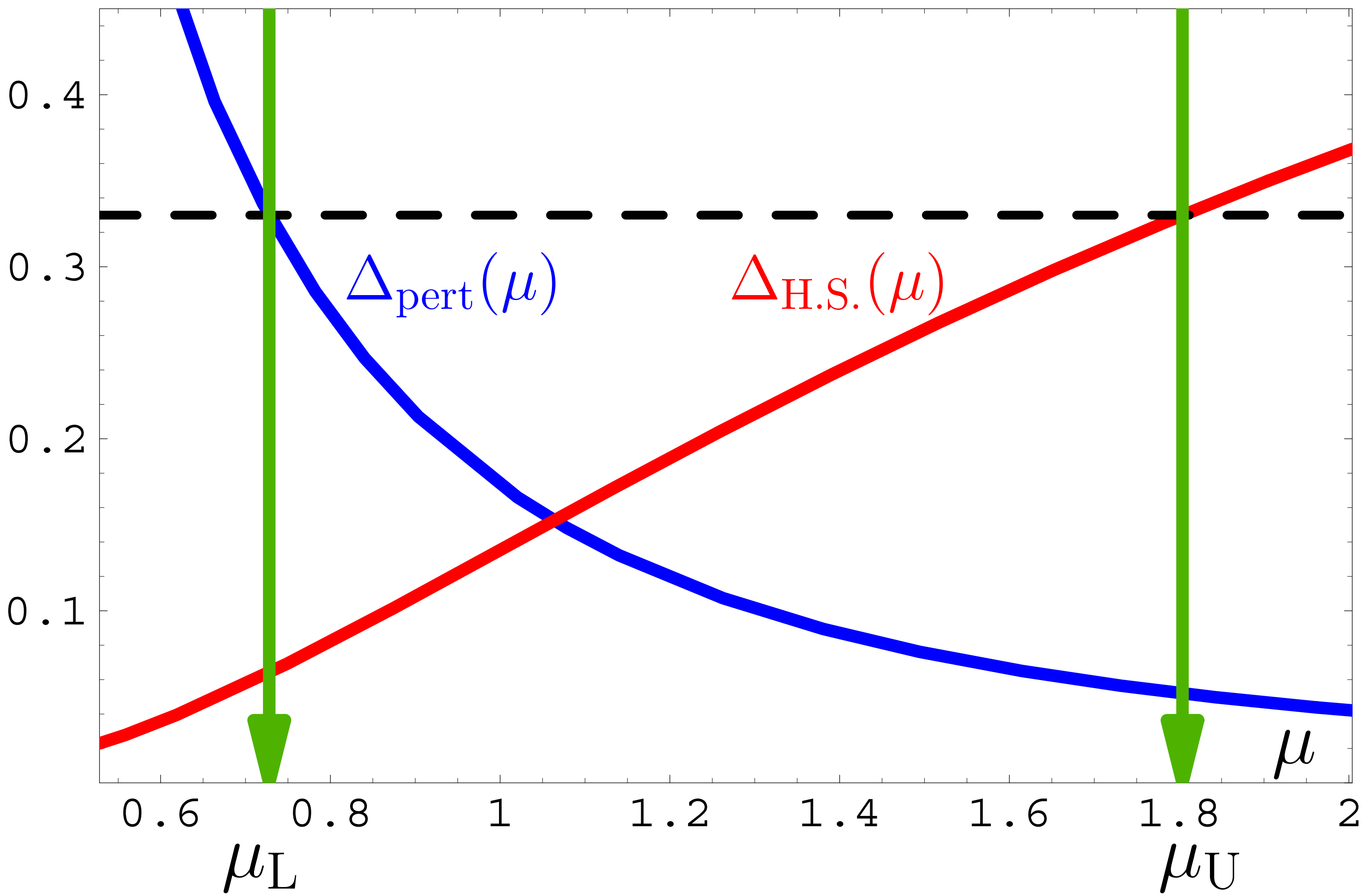
- Fidelity window: $\mu_{\mathrm{L}} \leq \mu \leq \mu_{\mathrm{U}}$. Only for μ inside it is reasonable to demand **minimal sensitivity** of SRs to variations in μ!

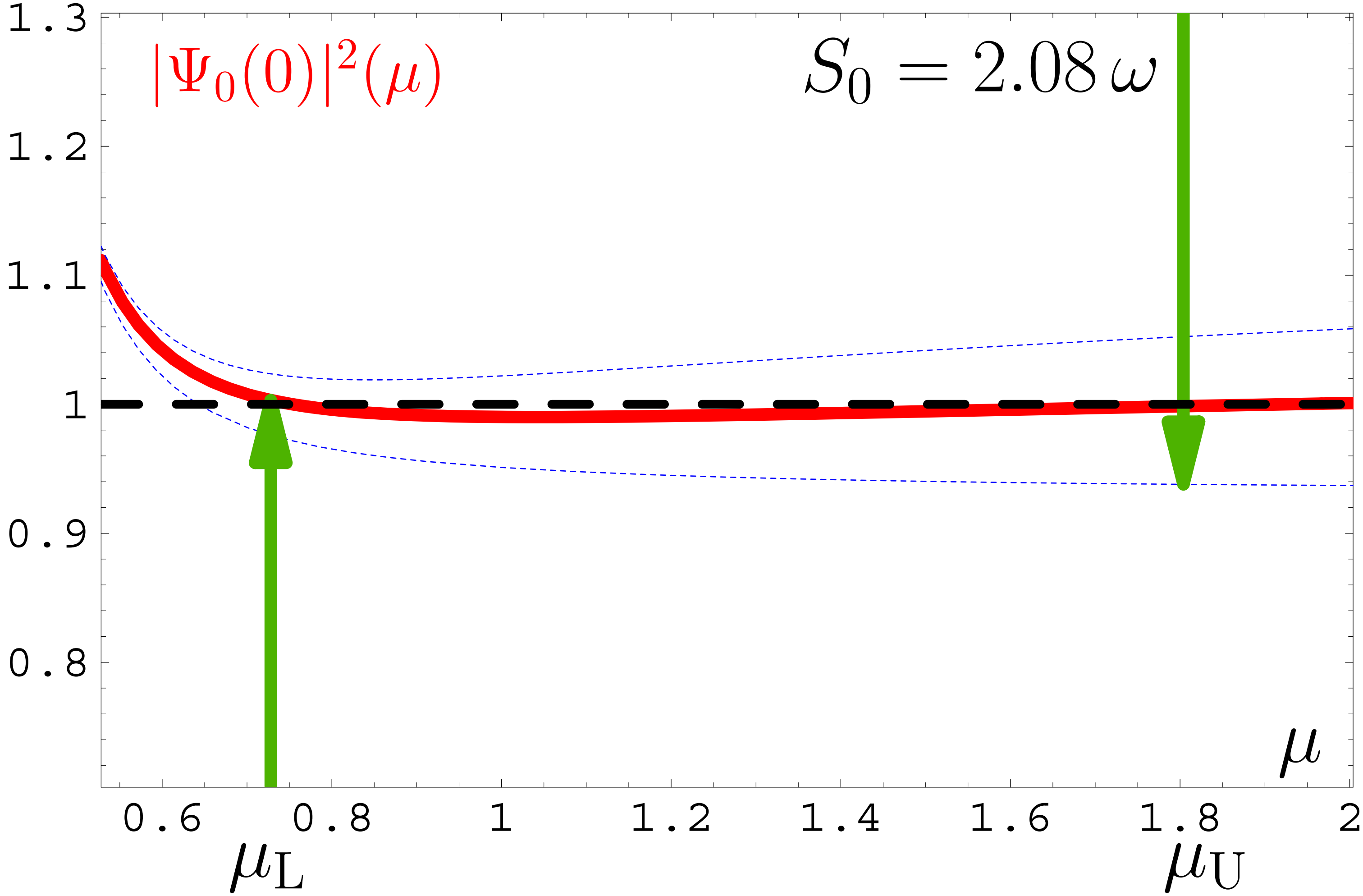
QM SRs: Setup with fixed $E_0 = 1$

We fix energy to the exact value $E_0 = 1$ and obtain fidelity window: $\mu_L = 0.73\,\omega$ and $\mu_U = 1.80\,\omega$



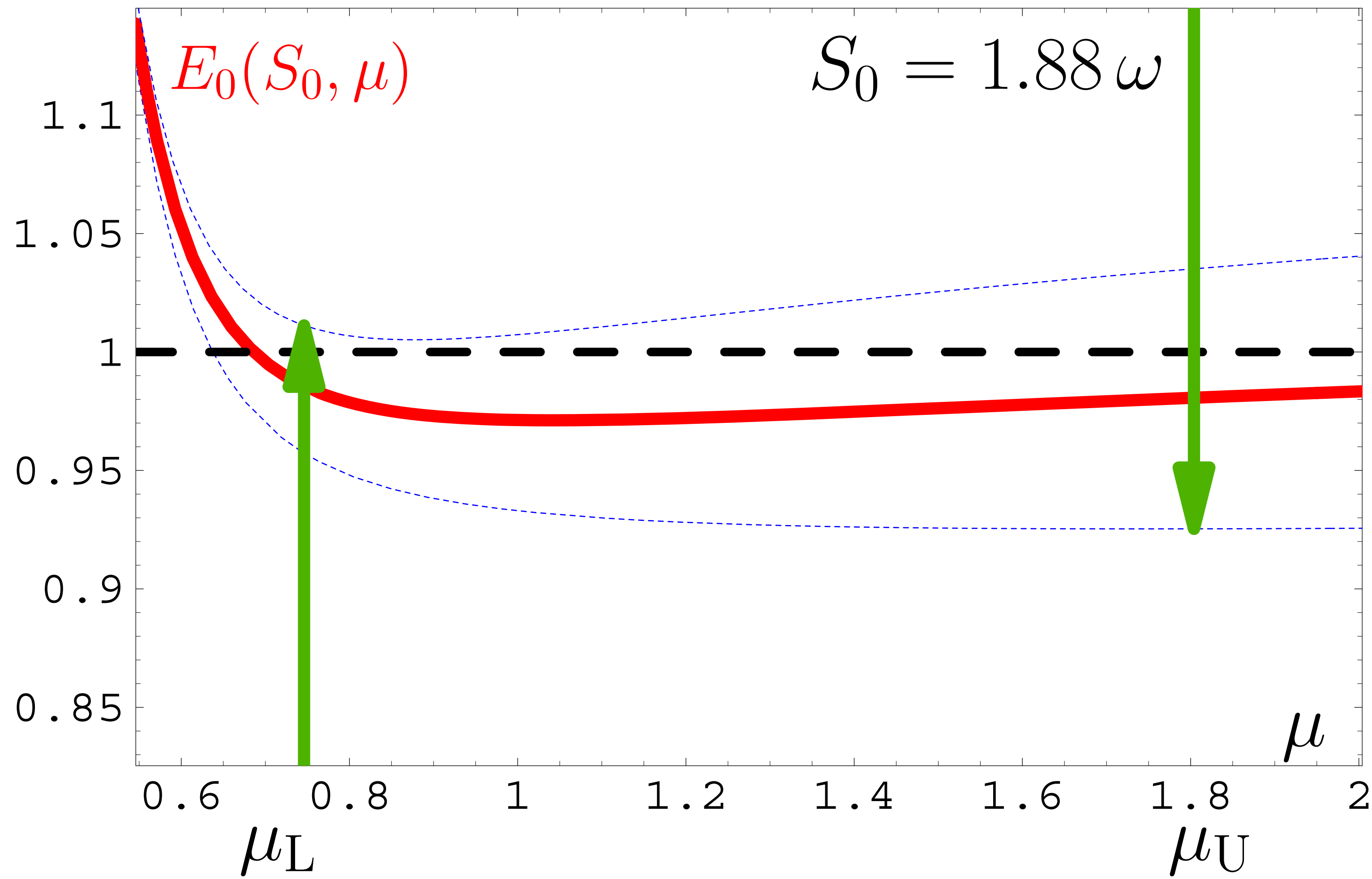
QM SRs: Setup with fixed $E_0 = 1$

We fix energy to the exact value $E_0 = 1$ and obtain $|\Psi_0(0)|^2 = 0.99$ with only 2 pow.corrs. (exact $|\Psi_0(0)|^2 = 1$)



QM SRs: Complete Setup

We take into account 3 power corrs. and obtain fidelity window $[0.74\,\omega; 1.8\,\omega]$ and $E_0 = 0.98\,\omega$ for $S_0 = 1.88\,\omega$:



QM SRs: Complete Setup

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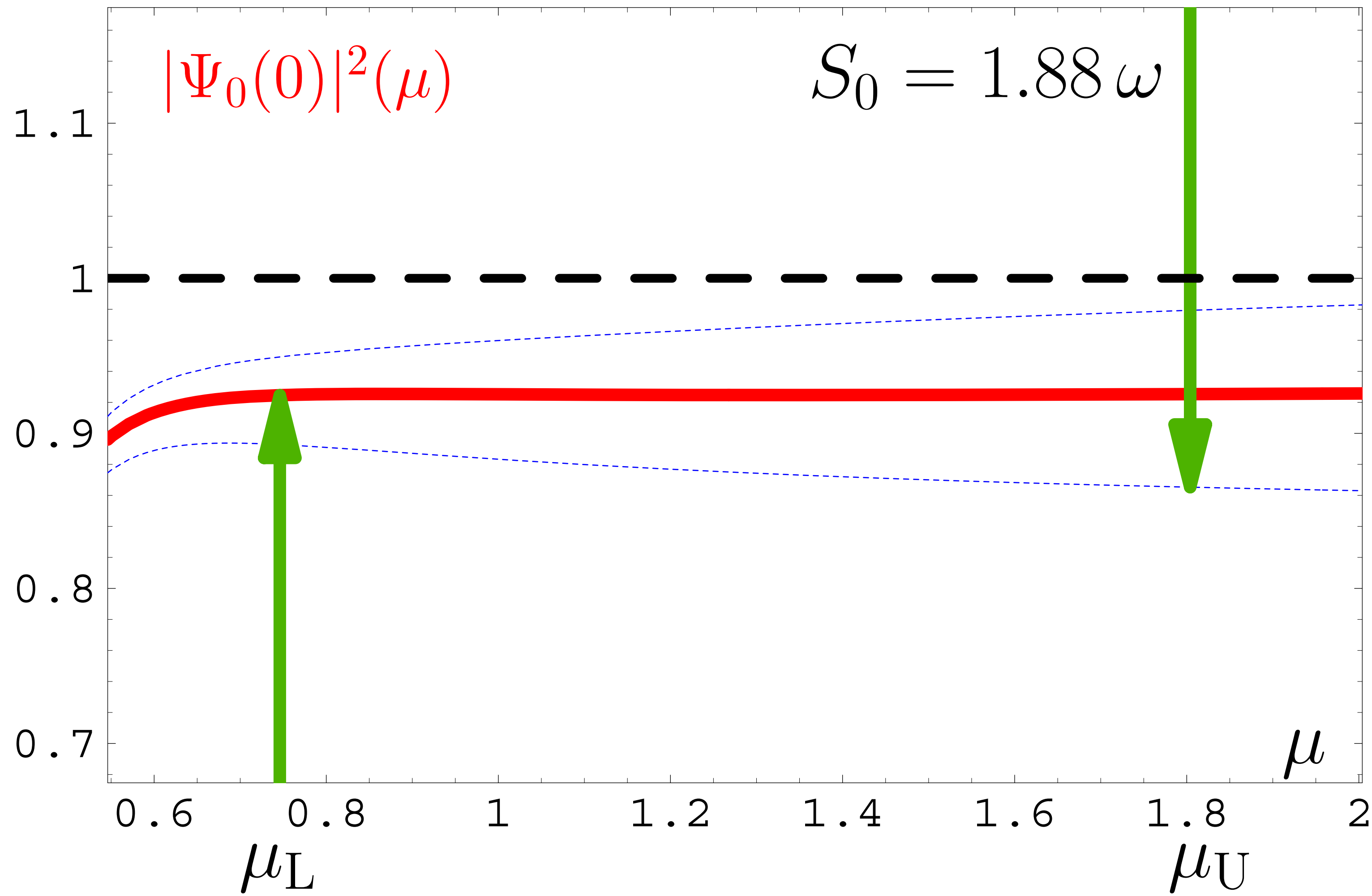

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- In **QCD** spectral density more close to perturbative!

QCD: Lagrangian, quarks and gluons

Gauge-invariant Lagrangian of QCD

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_{q=u,d,s,\dots} \bar{\psi}_q (i\hat{D} - m_q)\psi_q$$

contains only gluon ($G^a_{\mu\nu}(x)$) and quark ($\psi_q(x)$) fields.

QCD: Lagrangian, quarks and gluons

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These fields has color degrees of freedom: 3 for quarks $\psi_q^A(x)$ ($A = 1, 2, 3$) and 8 for gluons $G^a_{\mu\nu}(x)$ ($a = 1, \ldots, 8$).

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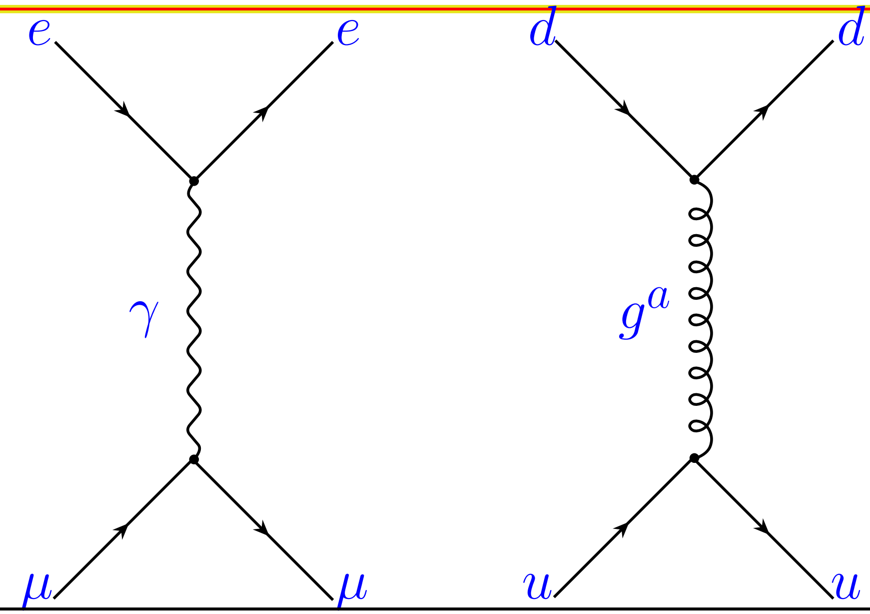
It is nonlinear due to **Non-Abelian** character ($f^{abc} \neq 0$).

QCD: Coloured gluons ⇒ Confinement

Consider $e\mu$- and qq-scattering (for d- and u-flavors): wavy line denotes **photon** and curved line – **gluon**.



Comparison:

Parameter	Photon	Gluon
Mass	0	0
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Vertex	$e\gamma_\mu$	$g_s\gamma_\mu(t^a)_{ij}$
Charge	0	$\neq 0$

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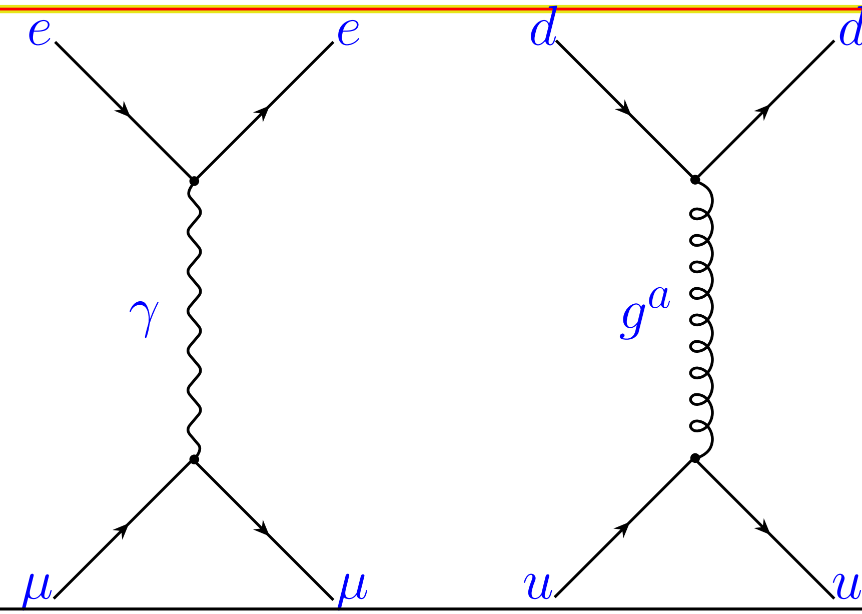

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Non-Abelian character of QCD ⇒ charged **gluons**.

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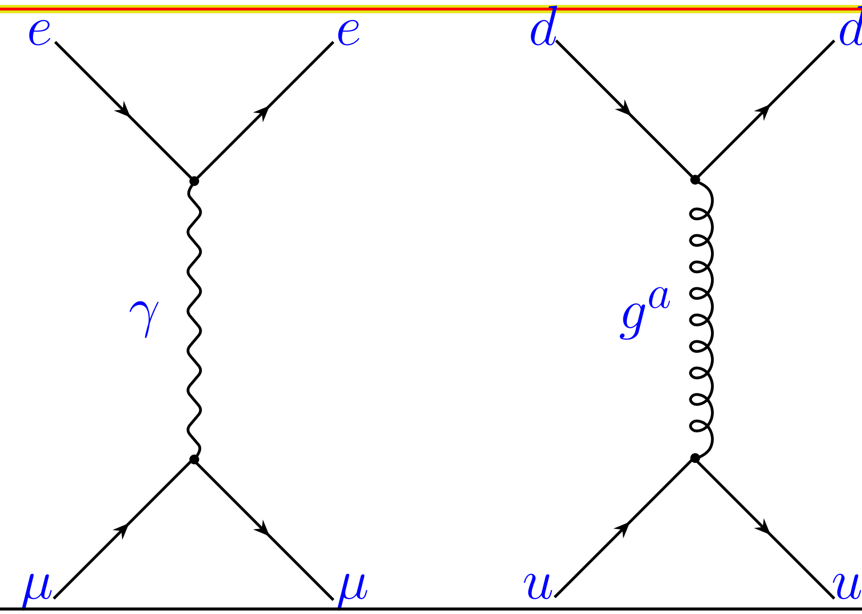

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Coloured **gluons** ⇒ **confinement!**

Massless QCD: What are Hadrons?

PS- and V-mesons composed of u- and d-quarks		
meson type	PS	V
composition	$\pi^0[\bar{u}u-\bar{d}d]$, $\pi^\pm[\bar{u}d, \bar{d}u]$	$\rho^0(\omega)[\bar{u}u-\bar{d}d]$, $\rho^\pm[\bar{u}d, \bar{d}u]$
mass	140 **MeV**	770(780) **MeV**

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Baryons composed of u- and d-quarks			
composition	$p[uud]$	$n[udd]$	$\Delta^{++}[uuu]$, $\Delta^{+}[uud]$, $\Delta^{0}[udd]$, $\Delta^{-}[ddd]$
mass	938 **MeV**	939 **MeV**	**1232 MeV**

QCD SRs:
Possibility to Study Hadrons
in Non-Perturbative QCD

QCD SRs: Hadrons in npQCD

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- In 1979 used to describe light hadrons in massless QCD.
- **Main idea**: to calculate correlators of hadron currents $\langle 0|T\left[J_1(x)J_2(0)\right]|0\rangle$ by two ways. Sum Rule is the result of matching.

QCD SRs: General scheme

Correlator of hadron currents via dispersion integral

$$F_{x\to q}\left[\langle 0|T\left[J_1(x)J_2(0)\right]|0\rangle\right](Q^2) \equiv \Pi(Q^2) =$$

$$= \int_0^\infty \frac{\rho_{12}(s)\ ds}{s+Q^2} + \text{“subtractions”}$$

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Apply Borel transform

$$B_{Q^2\to M^2}\left[\Pi(Q^2)\right] \equiv \Phi\left(M^2\right) = \int_0^\infty \rho_{12}(s)\ e^{-s/M^2}\frac{ds}{M^2}$$

to suppress “higher states” and to kill **“subtractions”** in DR.

QCD SRs: General scheme

1-st way: Operator Product Expansion with account for **quark and gluon condensates** in QCD vacuum

$$\Phi\left(Q^2\right) = \Phi_{\text{pert}}\left(Q^2\right) + c_{GG}\,\frac{\langle(\alpha_s/\pi)GG\rangle}{M^4} + c_{\bar{q}q}\,\frac{\alpha_s\langle\bar{q}q\rangle^2}{M^6}$$

Here $\langle\frac{\alpha_s}{\pi}G^a_{\mu\nu}G^{a\mu\nu}\rangle = 0.012$ **GeV**^{4}, $\alpha_s\langle\bar{q}q\rangle^2 = 0.0018$ **GeV**^{6}.

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2-nd way: phenomenological saturation of spectral density by hadronic states

$$\rho_{\text{had}}\left(s\right) = f_h^2\delta\left(s - m_h^2\right) + \rho_{\text{pert}}\left(s\right)\theta\left(s - s_0\right)$$

Our model is **ground state** h + **continuum**, which starts from threshold $s = s_0$.

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Borel transform is defined as $\Phi(M^2) =$

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Quark–Hadron Duality in QCD

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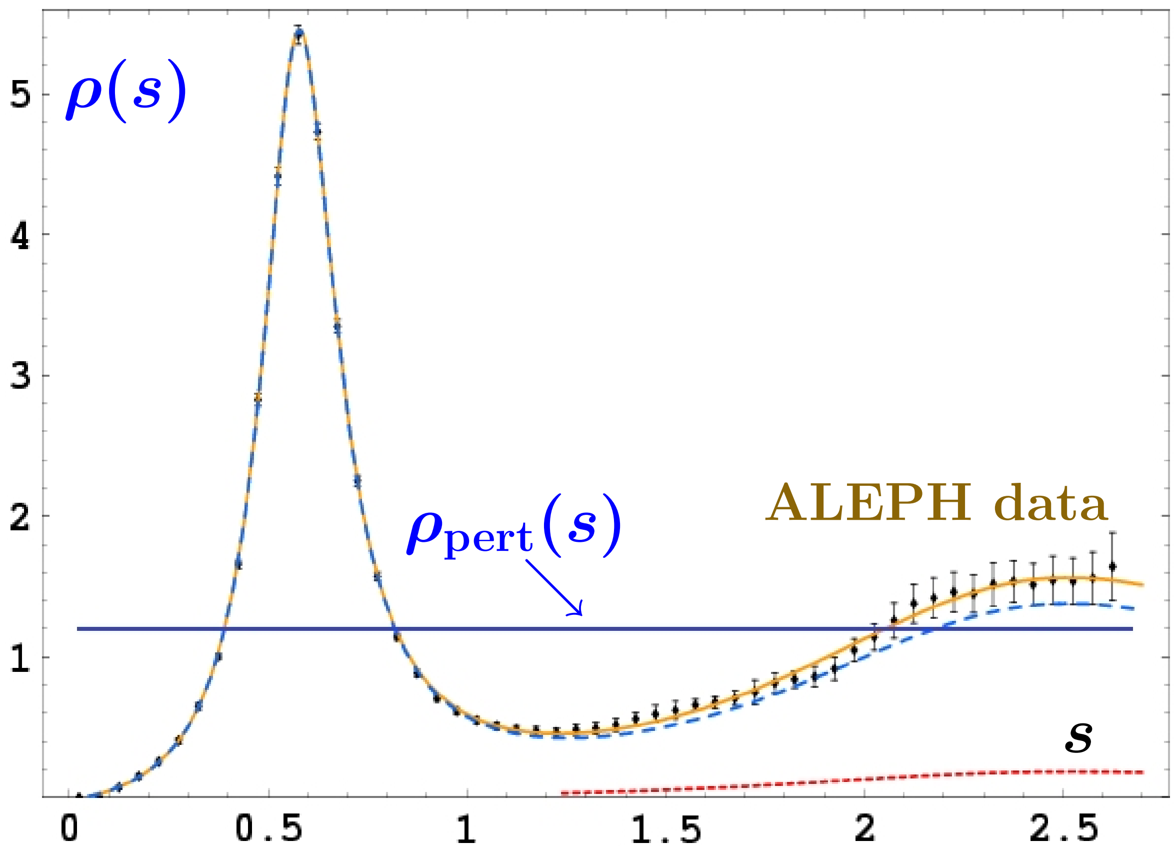

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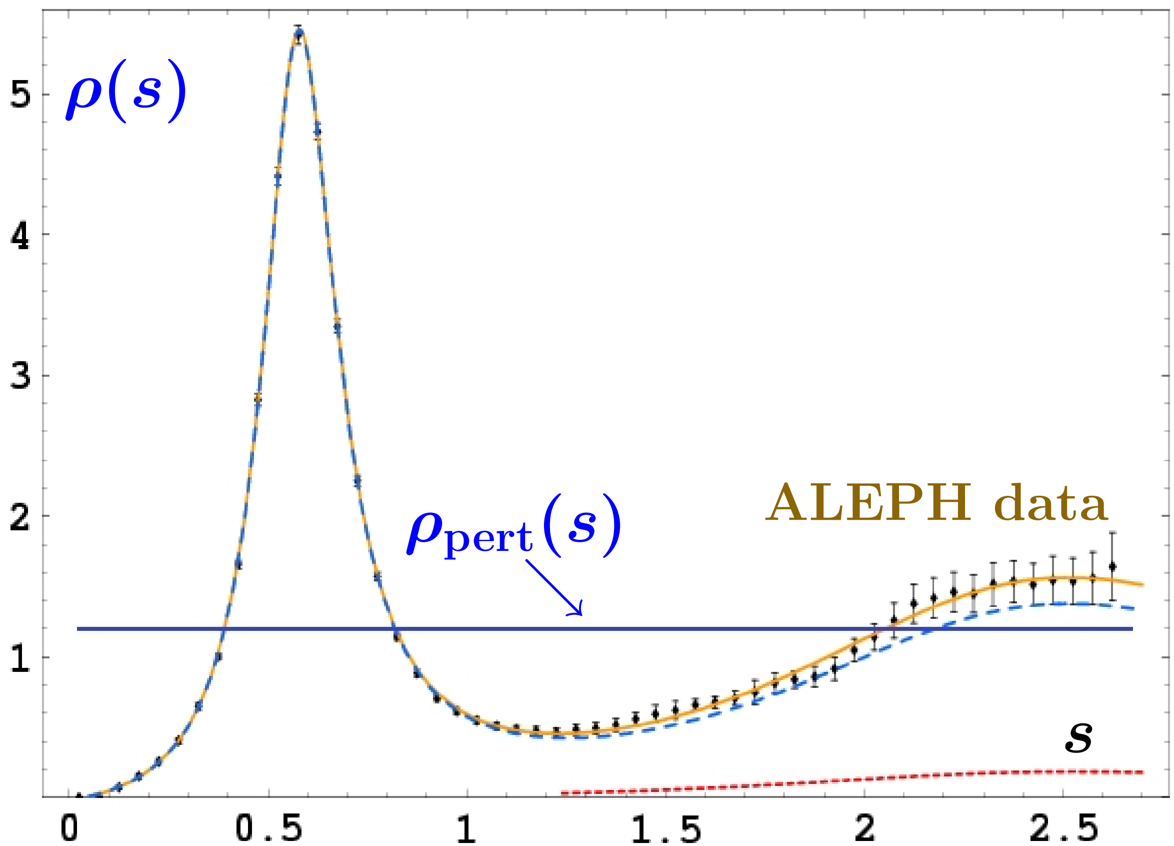

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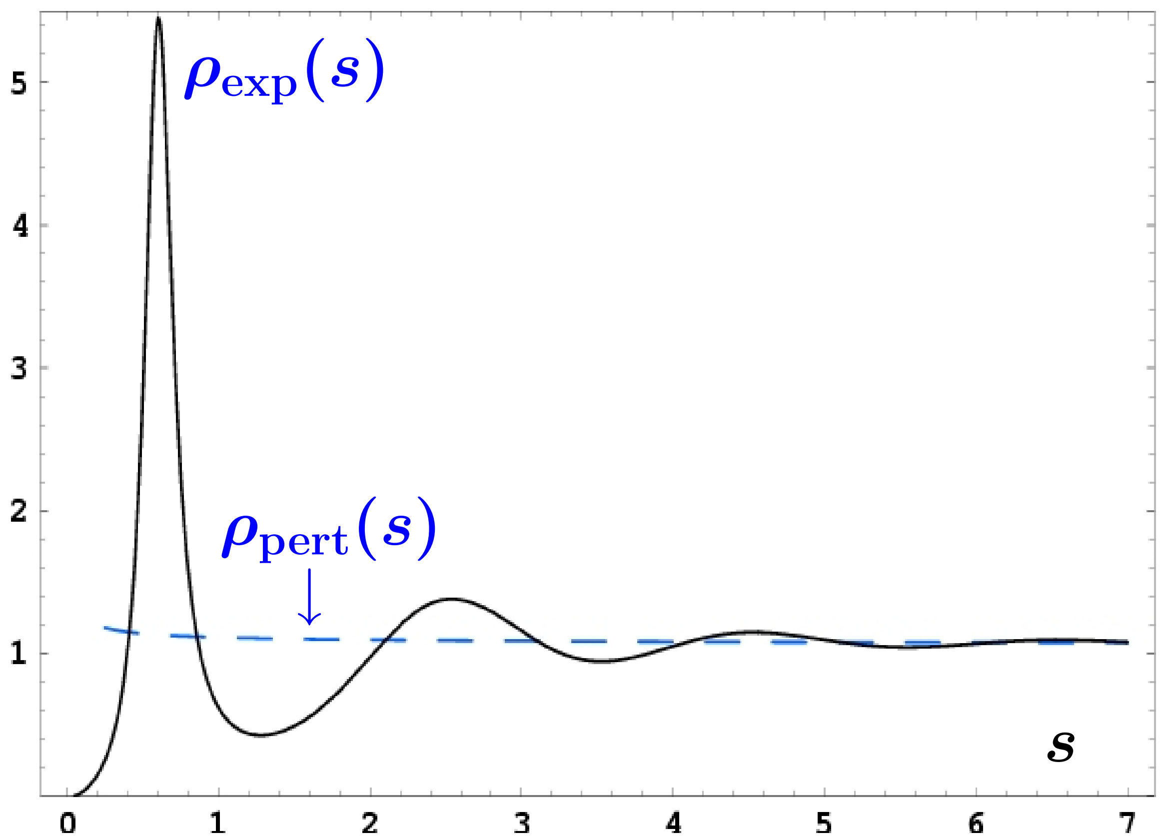

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3° Asymptotics starts at $s \geq 3$ **GeV**^{2}

QCD: Currents, Correlators and Spectral Densities of Real Particles

Currents related to π-mesons in QCD

Currents related to $\pi^{\pm}$ meson:

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AV: $J_{\mu 5}(x) = \bar{u}(x)\gamma_\mu\gamma_5 d(x)\,; \quad J^{\dagger}_{\mu 5}(x) = \bar{d}(x)\gamma_\mu\gamma_5 u(x)$

PS: $J_5(x) = i\,\bar{u}(x)\gamma_5 d(x)\,; \quad J^{\dagger}_5(x) = i\,\bar{d}(x)\gamma_5 u(x)$

Note that Dirac equation $i\,\hat{D}\,q(x) = m_q\,q(x)$ gives relation:

$$\partial^\mu J_{\mu 5}(x) = (m_u + m_d)\;J_5(x) \qquad (*)$$

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Decay constant f_π of physical pion $\pi(P)$ is defined via

$$\langle 0 \big| J_{\mu 5}(0) \big| \pi(P) \rangle = i\,f_\pi\,P_\mu$$

It was measured in decay $\pi \to \mu\nu_\mu$ to be $f_\pi = 132$ **MeV**.

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Eq. **(*)** then gives $\langle 0|J_5(0)|\pi(P)\rangle = \dfrac{f_\pi\,m_\pi^2}{m_u + m_d}$.

Currents related to vector mesons in QCD

Currents related to $\rho^{\pm}$ meson:

$$J_\mu(x) = \bar{u}(x)\gamma_\mu d(x)\,; \quad J^{\dagger}{}_\mu(x) = \bar{d}(x)\gamma_\mu u(x)$$

Decay constant f_ρ of physical $\rho^{\pm}(P,\varepsilon)$-meson with polarization ε and momentum P, satisfying $(P\,\varepsilon) = 0$ and $(\varepsilon,\varepsilon) = -1$,

$$\langle 0 \big| J_\mu(0) \big| \rho(P,\varepsilon)\rangle = f_\rho\, m_\rho\, \varepsilon_\mu$$

Decay $\rho^0 \to e^+e^-$ allows to measure $f_{\rho^0} = 150$ **MeV**, that gives $f_{\rho^{\pm}} = 210$ **MeV**.

Vector current correlator $\Pi_{\mu\nu}$

Lorentz invariance and vector current conservation dictate

$$\Pi_{\mu\nu}(q) = i\int d^4x\, e^{iqx}\langle 0|T\left[J^{\mu}(x)J_{\nu}(0)\right]|0\rangle = \left[q_{\mu}\,q_{\nu} - g_{\mu\nu}\,q^2\right]\Pi($$

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Question: What is dimensionality of $\Pi(q)$?

$$\begin{aligned} D[J^\mu(x)] &= D[\psi]^2 = M^3 \\ D[\Pi_{\mu\nu}(q)] &= D[J^\mu(x)]^2 - M^4 = M^2 \\ D[\Pi(q)] &= ? \end{aligned}$$

Vector current correlator $\Pi_{\mu\nu}$

Lorentz invariance and vector current conservation dictate

Inserting $\hat{1}$ in between currents we obtain

$$\Pi(q) = \frac{-i}{3q^2} \sum_{X(p)} \int_0^{\infty} dt\, e^{iq_0 t} \int d^3\vec{x}\, e^{-i\vec{q}\vec{x}} \langle 0 \big| J^{\mu}(x) \big| X(p) \rangle \langle X(p) \big| J^{\dagger}_{\mu}(0) \big| 0 \rangle$$

$$+ \frac{-i}{3q^2} \sum_{X(p)} \int_{-\infty}^{0} dt\, e^{iq_0 t} \int d^3\vec{x}\, e^{-i\vec{q}\vec{x}} \langle 0 \big| J^{\dagger}_{\mu}(0) \big| X(p) \rangle \langle X(p) \big| J^{\mu}(x) \big| 0 \rangle$$

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$$= \frac{-i\,(2\pi)^3}{3q^2} \sum_{X(p)} \delta(\vec{p} - \vec{q})\, \theta(p_0) \left| \langle 0 | J_{\mu}(0) | X(p) \rangle \right|^2$$

$$\times \int_0^{\infty} dt \left[e^{i(q_0 - p_0)t} + e^{-i(q_0 + p_0)t} \right]$$

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Then $\Pi(q^2) = \frac{-i\,(2\pi)^3}{3q^2} \sum_{X(p)} \delta(\vec{p} - \vec{q}) \left| \langle 0 | J_\mu(0) | X(p) \rangle \right|^2 \times$

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We have the following identities

$$\int_0^\infty dt\, e^{\pm i\alpha t} = \pi\, \delta(\alpha) \pm i\, \mathcal{P} \frac{1}{\alpha}$$

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After all substitutions:

$$\mathbf{Im}\Pi(q^2) = -\pi\,\frac{(2\pi)^3}{3q^2} \sum_{X(p)} \delta(\vec{p}-\vec{q})\delta(p_0 - |q_0|) \Big| \langle 0 \big| J_\mu(0) \big| X(p) \rangle \Big|^2$$

Vector current correlator $\Pi_{\mu\nu}$

So, we have $\boxed{\frac{1}{\pi}\,\mathbf{Im}\,\Pi(q^2) = \rho(q^2)\theta(|q_0|) = \rho(q^2)}$ with

$$\rho(q^2)\theta(q_0) = \frac{-(2\pi)^3}{3q^2}\sum_{X(p)}\delta^{(4)}(q-p)\theta(p_0)\Big|\langle 0|J_\mu(0)|X(p)\rangle\Big|^2$$

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Lorentz invariance dictates

$$\langle 0|J^\mu(x)|X(p)\rangle = [A\,p_\mu + B\,\varepsilon_\mu]\,e^{-ipx}$$

with $p\cdot\varepsilon = 0$, and therefore $\varepsilon\cdot\varepsilon = -1$. From current conservation it follows $A = 0$, i. e. ($B = f_X\,m_X$)

$$\langle 0|J^\mu(x)|X(p)\rangle\langle X(p)|J^\dagger_\mu(x)|0\rangle = |f_X|^2\,m_X^2\varepsilon^2 \leq 0$$

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$$\boxed{\rho(s) = \sum_X \big|f_X\big|^2\delta(s-m_X^2) \geq 0}$$

Spectral density of correlators $\Pi_{\mu\nu}$ *and* $\Pi^{+}_{\mu\nu}$

So, we have $$\frac{1}{\pi}\,\mathrm{Im}\,\Pi(q^2) = \rho(q^2)\theta(|q_0|) = \rho(q^2)$$

If we consider correlator

$$\Pi^{+}_{\mu\nu}(q) = i\int d^4x\, e^{iqx}\langle 0|J^{\mu}(x)J_{\nu}(0)|0\rangle = \left[q_{\mu}q_{\nu} - g_{\mu\nu}q^2\right]\Pi^{+}(q$$

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Now we can say why we put T-product in correlators
– then spectral densities, defined only by **real particles**,
are **Lorentz invariant** and **depend only on q^2** !

Relation with cross section $e^+e^- \longrightarrow hadrons$

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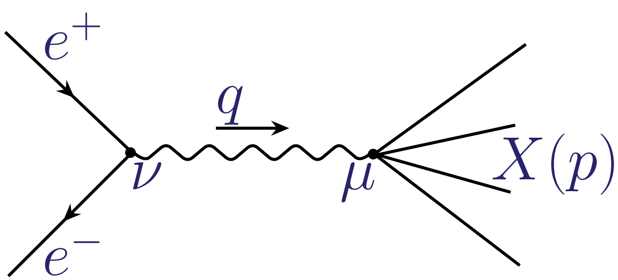
Important! This function naturally appears in 1-photon QED description of $e^+e^- \longrightarrow$ hadrons:



$$\bar{u}(k)\gamma_\mu u(k')\frac{ie^2}{q^2}\langle X(p)|J_\mu(q)|0\rangle$$

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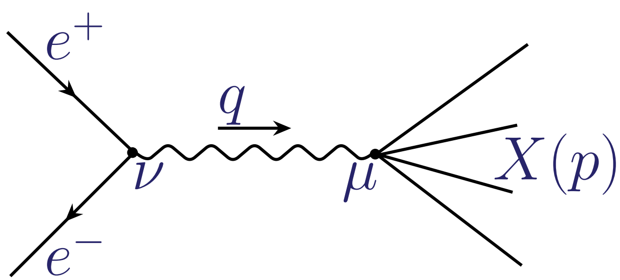
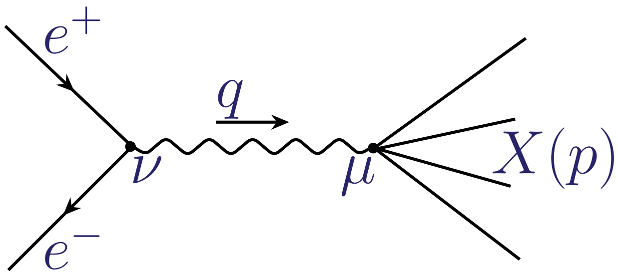


$$\boxed{\sigma_{\text{had}}(s) = \frac{16\,\pi^3\,\alpha^2}{s}\rho(s)}$$

Relation with cross section $e^+e^- \longrightarrow$ *hadrons*

In 1-photon approximation of QED:



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Here we explicitly extracted as a factor cross-section $\sigma_{\mu^+\mu^-}(s) = 4\pi\alpha^2/(3\,s)$ of the process $e^+e^- \to \mu^+\mu^-$.

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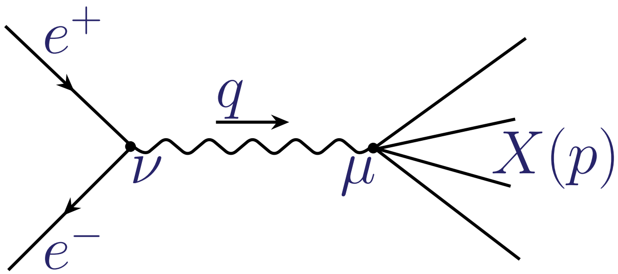

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$$R(s) = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}\,; \quad \rho(s) = \frac{R(s)}{12\pi^2}$$

Relation with cross section $e^+e^- \longrightarrow$ *hadrons*

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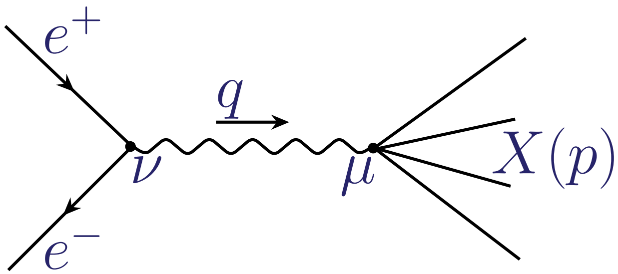

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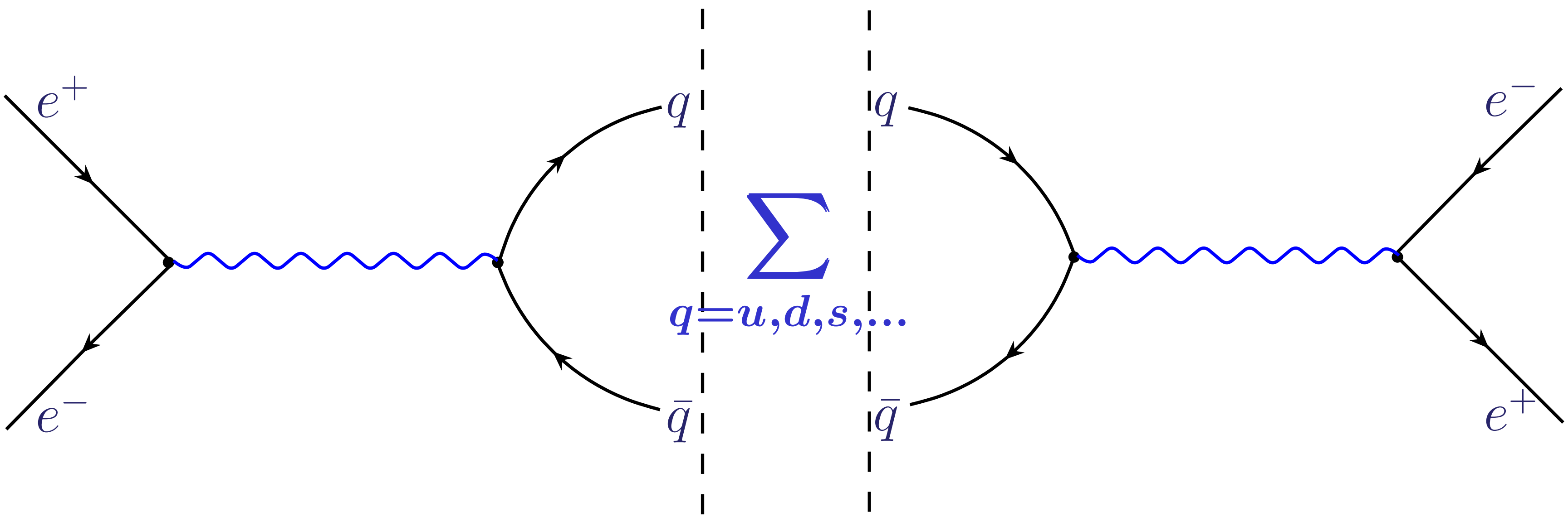

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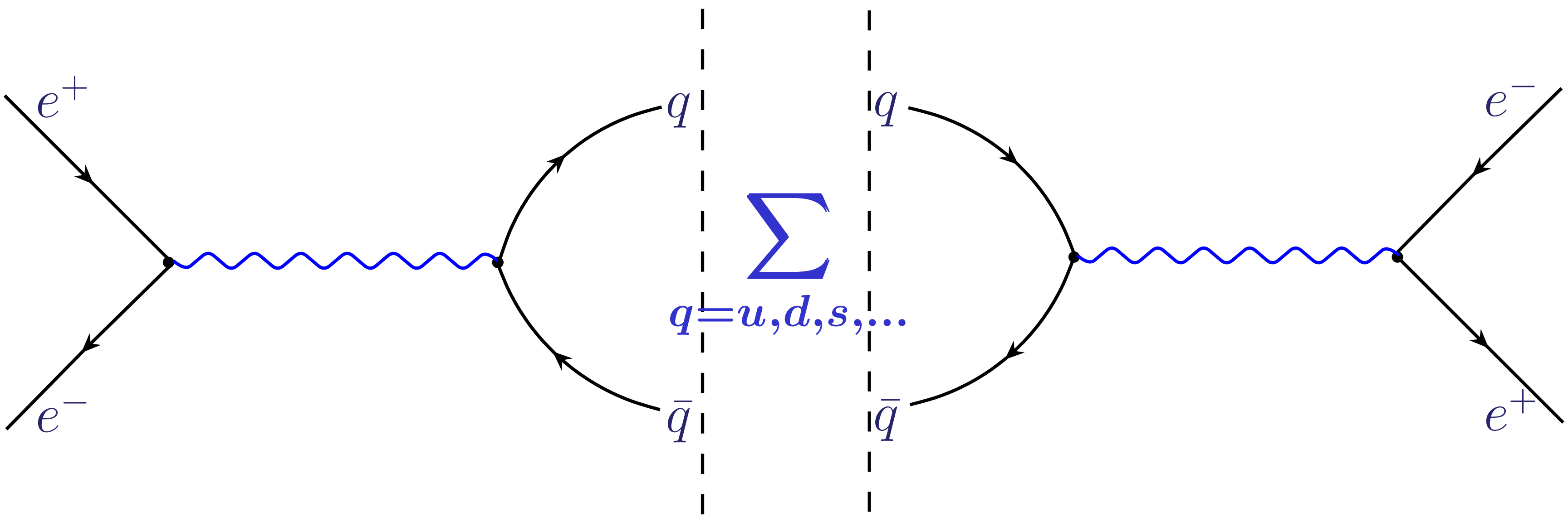

$$R^{\text{tree}}_{\text{QCD}}(s \le m_c^2) = ?$$

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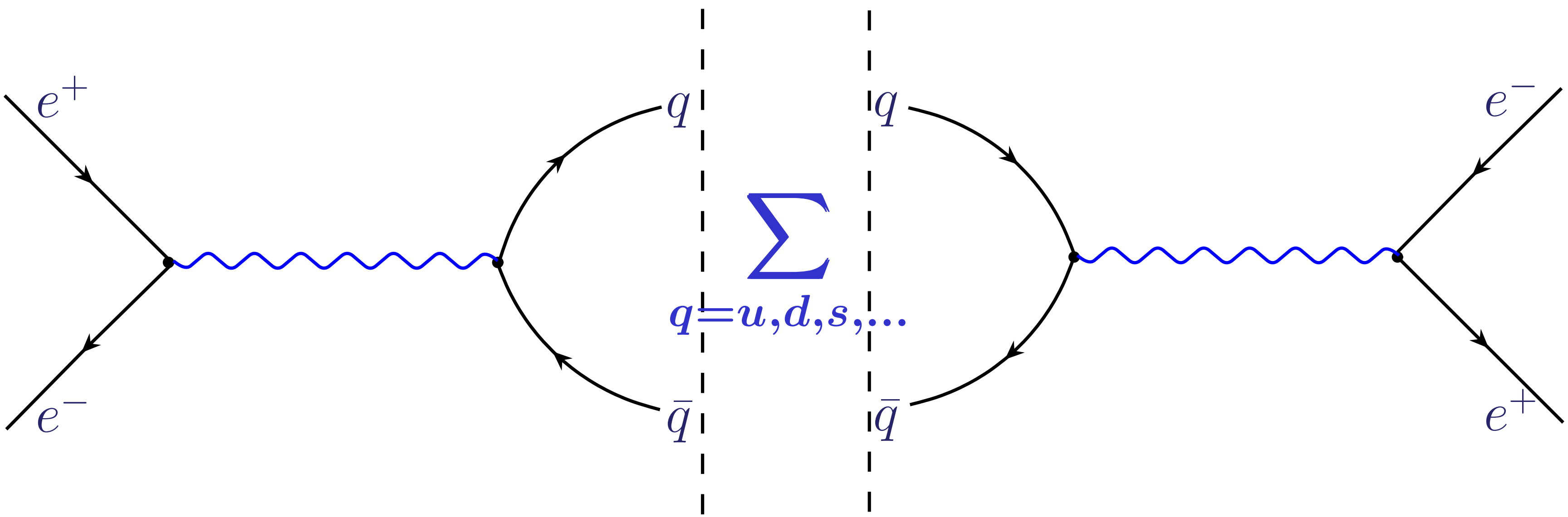

$$R^{\text{tree}}_{\text{QCD}}(s \leq m_c^2) = N_c \sum_{q=u,d,s} e_q^2 =?$$

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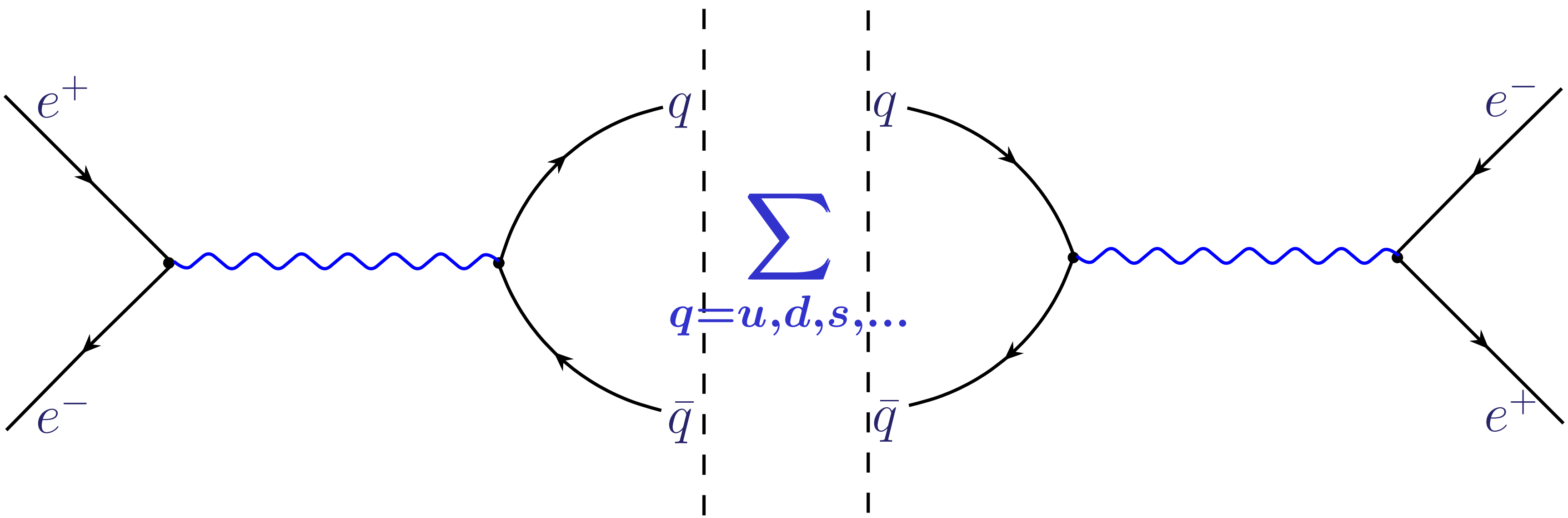

$$R^{\text{tree}}_{\text{QCD}}(s \le m_c^2) = N_c \sum_{q=u,d,s} e_q^2 = 3\left(\frac{1}{9} + \frac{1}{9} + \frac{4}{9}\right) = 2$$